# FeenoX annotated examples

## Contents

All the files needed to run theses examples are available in the examples directory of the Git repository and any of the tarballs, both source and binary.

For a presentation of these examples, see the “FeenoX Overview Presentation” (August 2021).

- Recording (audio in Spanish, slides in English)
- Slides in PDF
- Markdown examples sources

Also, the `tests` directory in the Github repository has dozens of test cases which can be used as examples for reference and for mathematical verification of the results obtained with FeenoX.

## 1 Hello World (and Universe)!

```
---
intro: |
  # Hello World (and Universe)!
...
PRINT "Hello $1!"
---
terminal: |
  $ feenox hello.fee World
  Hello World!
  $ feenox hello.fee Universe
  Hello Universe!
  $
...
```

```
```

```
PHASE_SPACE x y z     # Lorenz 'attractors phase space is x-y-z
end_time = 40         # we go from t=0 to 40 non-dimensional units

sigma = 10            # the original parameters from the 1963 paper
r = 28
b = 8/3

x_0 = -11             # initial conditions
y_0 = -16
z_0 = 22.5

# the dynamical system's equations written as naturally as possible
x_dot = sigma*(y - x)
y_dot = x*(r - z) - y
z_dot = x*y - b*z

PRINT t x y z        # four-column plain-ASCII output
```

```
$ feenox lorenz.fee > lorenz.dat
$ gnuplot lorenz.gp
$ python3 lorenz.py
$ sh lorenz2x3d.sh < lorenz.dat > lorenz.html
```

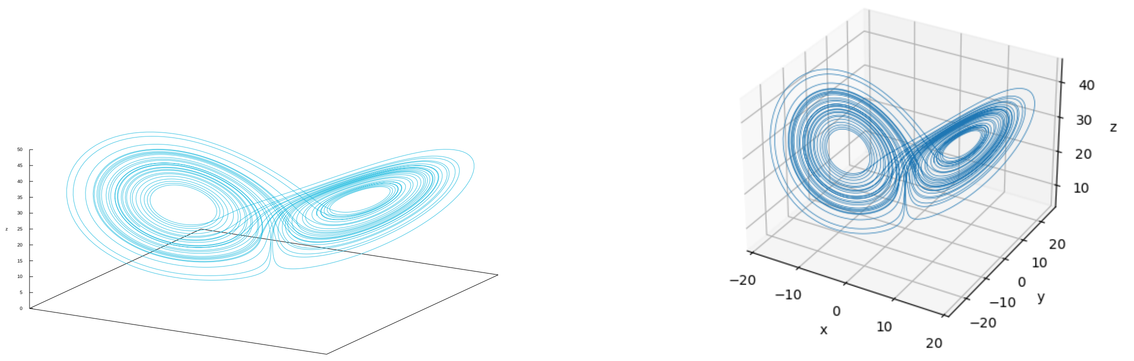


Figure 1: The Lorenz attractor computed with FeenoX plotted with two different tools

```
DEFAULT_ARGUMENT_VALUE 1 2.6   # by default show r in [2.6:4]
DEFAULT_ARGUMENT_VALUE 2 4

steps_per_r = 2^10
steps_asymptotic = 2^8
steps_for_r = 2^10

static_steps = steps_for_r*steps_per_r

# change r every steps_per_r steps
IF mod(step_static,steps_per_r)=1
 r = quasi_random($1,$2)
ENDIF

x_init = 0.5             # start at x = 0.5
x = r*x*(1-x)            # apply the map

IF step_static-steps_per_r*floor(step_static/steps_per_r)>(steps_per_r-steps_asymptotic)
 # write the asymptotic behavior only
 PRINT r x
ENDIF
```

```
$ gnuplot
gnuplot> plot "< feenox logistic.fee" w p pt 50 ps 0.02
gnuplot> quit
$
```

```
PROBLEM thermal 1D     # tell FeenoX what we want to solve
READ_MESH slab.msh     # read mesh in Gmsh's v4.1 format
k = 1                  # set uniform conductivity
BC left  T=0           # set fixed temperatures as BCs
BC right T=1           # ”left” and ”right” are defined in the mesh
SOLVE_PROBLEM          # we are ready to solve the problem
PRINT T(1/2)           # ask for the temperature at x=1/2
```

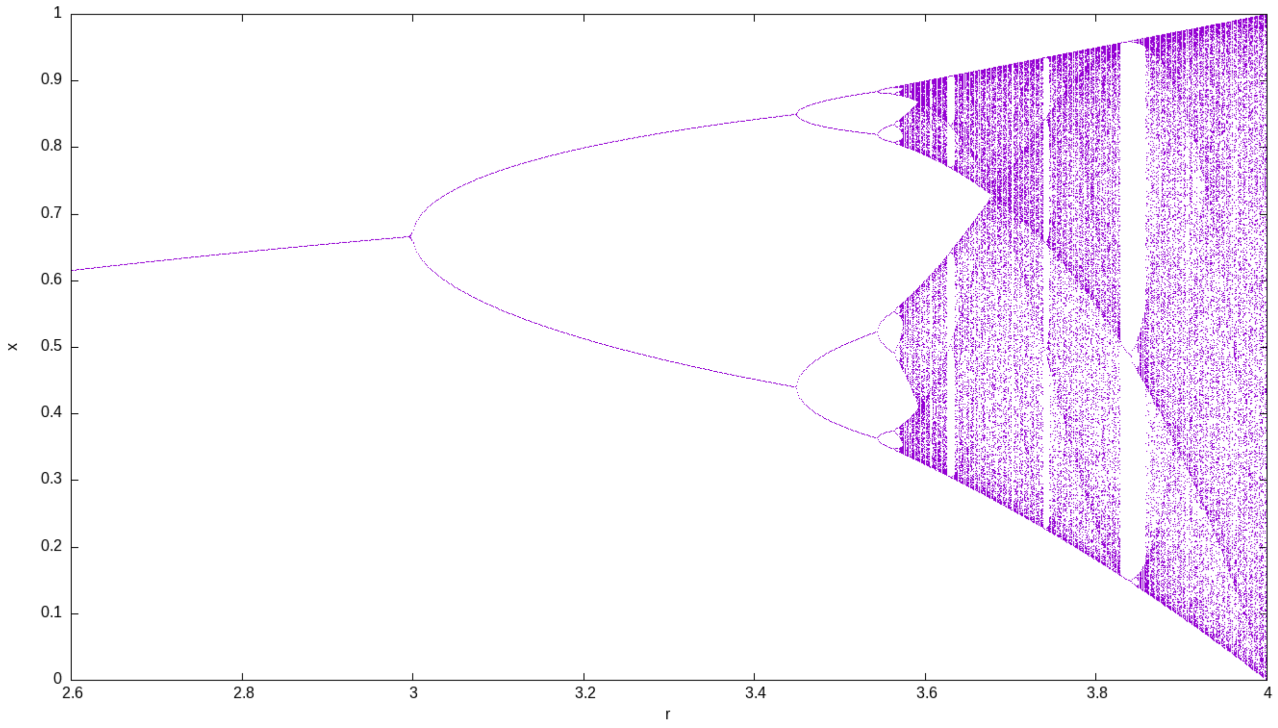


Figure 2: Asymptotic behavior of the logistic map.

```
$ gmsh -1 slab.geo
[...]
Info    : 4 nodes 5 elements
Info    : Writing 'slab.msh'...
[...]
$ feenox thermal-1d-dirichlet-uniform-k.fee
0.5
$
```

```
# NAFEMS Benchmark LE-10: thick plate pressure
PROBLEM mechanical 3D
READ_MESH nafems-le10.msh   # mesh in millimeters

# LOADING: uniform normal pressure on the upper surface
BC upper    p=1      # 1 Mpa

# BOUNDARY CONDITIONS:
BC DCD'C'   v=0      # Face DCD'C' zero y-displacement
BC ABA'B'   u=0      # Face ABA'B' zero x-displacement
BC BCB'C'   u=0 v=0  # Face BCB'C' x and y displ. fixed
BC midplane w=0      #  z displacements fixed along mid-plane

# MATERIAL PROPERTIES: isotropic single-material properties
E = 210e3   # Young modulus in MPa
nu = 0.3    # Poisson's ratio
```

```
SOLVE_PROBLEM   # solve!

# print the direct stress y at D (and nothing more)
PRINT "sigma_y @ D = " sigmay(2000,0,300) "MPa"

# write post-processing data for paraview
WRITE_MESH nafems-le10.vtk sigmay VECTOR u v w
```

```
$ gmsh -3 nafems-le10.geo
[...]
$ feenox nafems-le10.fee
sigma_y @ D =   -5.38016        MPa
$
```

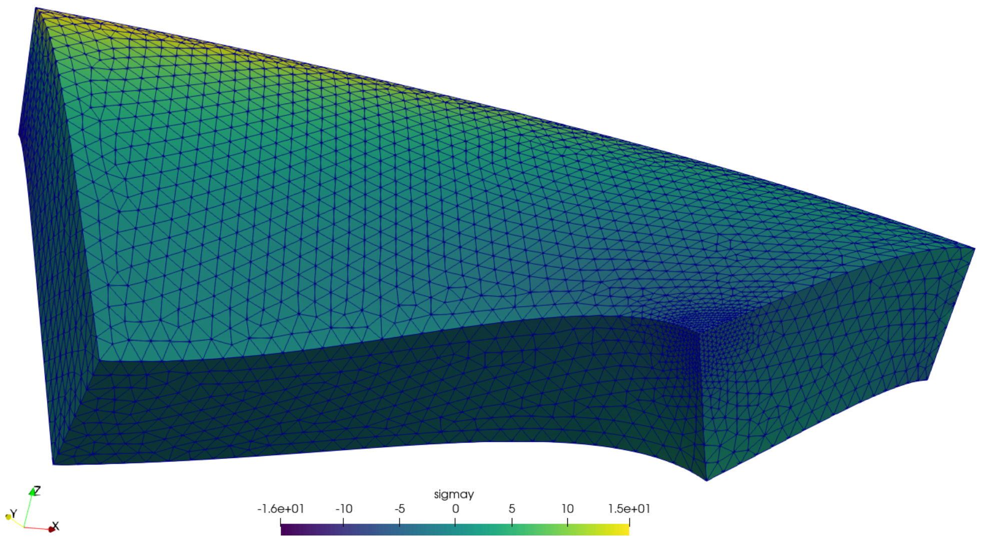


Figure 3: Normal stress $\sigma_y$ refined around point $D$ over 5,000x-warped displacements for LE10 created with Paraview

```
# NAFEMS Benchmark LE-11: solid cylinder/taper/sphere-temperature
PROBLEM mechanical 3D
READ_MESH nafems-le11.msh

# linear temperature gradient in the radial and axial direction
# as an algebraic expression as human-friendly as it can be
T(x,y,z) := sqrt(x^2 + y^2) + z

BC xz     v=0       # displacement vector is [u,v,w]
BC yz     u=0       # u = displacement in x
BC xy     w=0       # v = displacement in y
BC HIH'I' w=0       # w = displacement in z

E = 210e3*1e6       # mesh is in meters, so E=210e3 MPa -> Pa
nu = 0.3            # dimensionless
alpha = 2.3e-4      # in 1/ºC as in the problem
SOLVE_PROBLEM

# for post-processing in Paraview
WRITE_MESH nafems-le11.vtk VECTOR u v w   T sigmax sigmay sigmaz
```

```
PRINT "sigma_z(A) =" %.2f sigmaz(1,0,0)/1e6 "MPa" SEP " "
PRINT "wall time  =" %.2f wall_time() "seconds"  SEP " "
```

```
$ gmsh -3 nafems-le11.geo
[...]
$ feenox nafems-le11.fee
sigma_z(A) = -105.04 MPa
wall time  = wall time  = 1.91 seconds
$
```

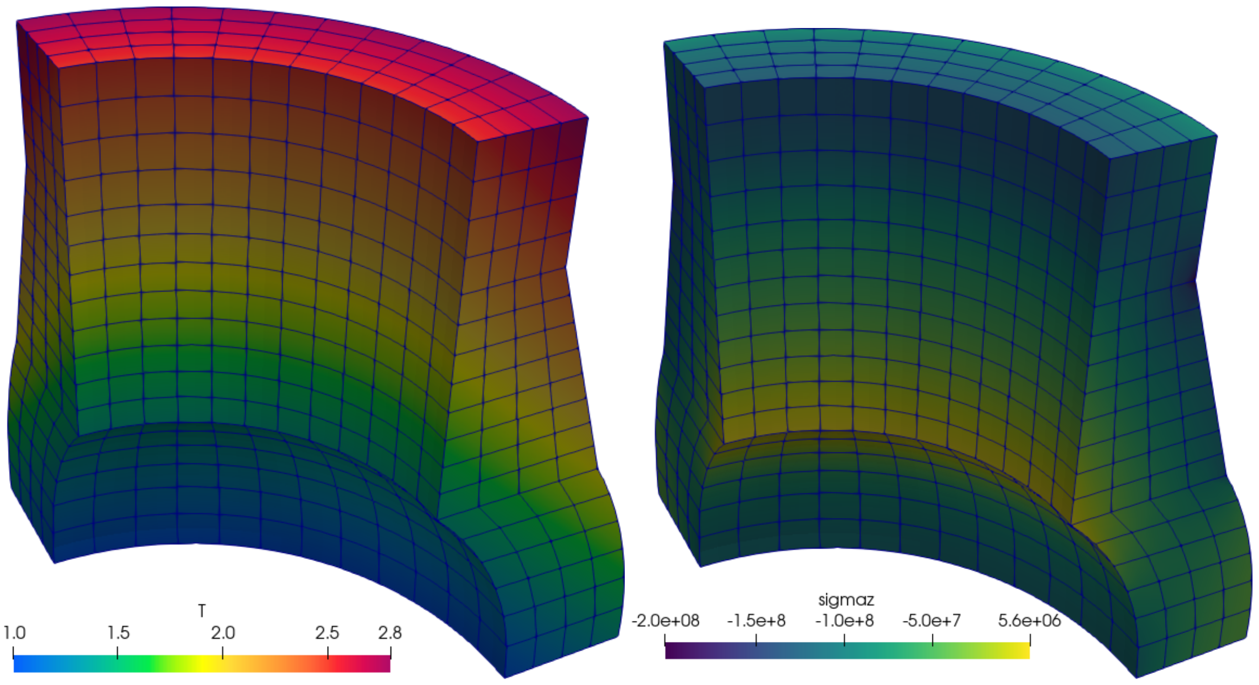


Figure 4: The NAFEMS LE11 problem results

```
PROBLEM mechanical plane_stress
READ_MESH nafems-le1.msh

E = 210e3
nu = 0.3

BC AB u=0
BC CD v=0
BC BC tension=10

SOLVE_PROBLEM

WRITE_MESH nafems-le1.vtk VECTOR u v 0 sigmax sigmay tauxy
PRINT "σy at point D = " %.4f sigmay(2000,0) "(reference is 92.7)" SEP " "
```

```
$ gmsh -2 nafems-le11.geo
```

```
[...]
$ feenox nafems-le1.fee
σy at point D =  92.7011 (reference is 92.7)
$
```

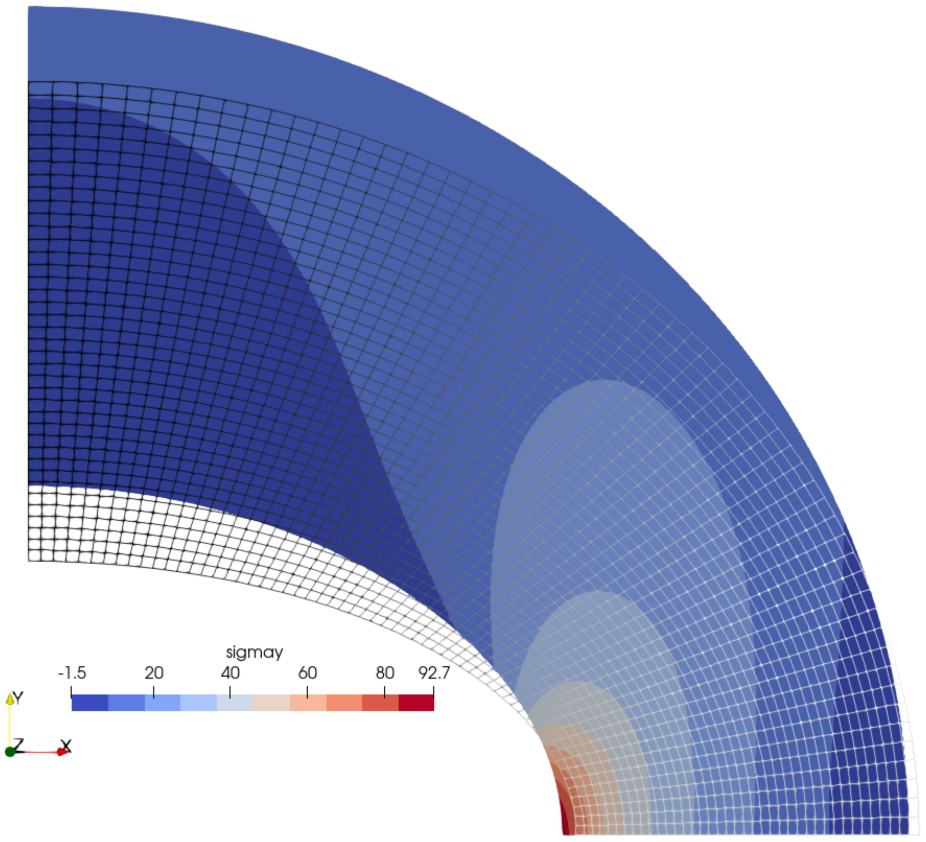


Figure 5: Normal stress $\sigma_y$ over 500x-warped displacements for LE1 created with Paraview

```
PROBLEM laplace 2D  # pretty self-descriptive, isn't it?
READ_MESH maze.msh

# boundary conditions (default is homogeneous Neumann)
BC start  phi=0
BC end    phi=1

SOLVE_PROBLEM

# write the norm of gradient as a scalar field
# and the gradient as a 2d vector into a .msh file
WRITE_MESH maze-solved.msh \
    sqrt(dphidx(x,y)^2+dphidy(x,y)^2) \
    VECTOR dphidx dphidy 0
```

```
$ gmsh -2 maze.geo
```

```
[...]
$ feenox maze.fee
$
```

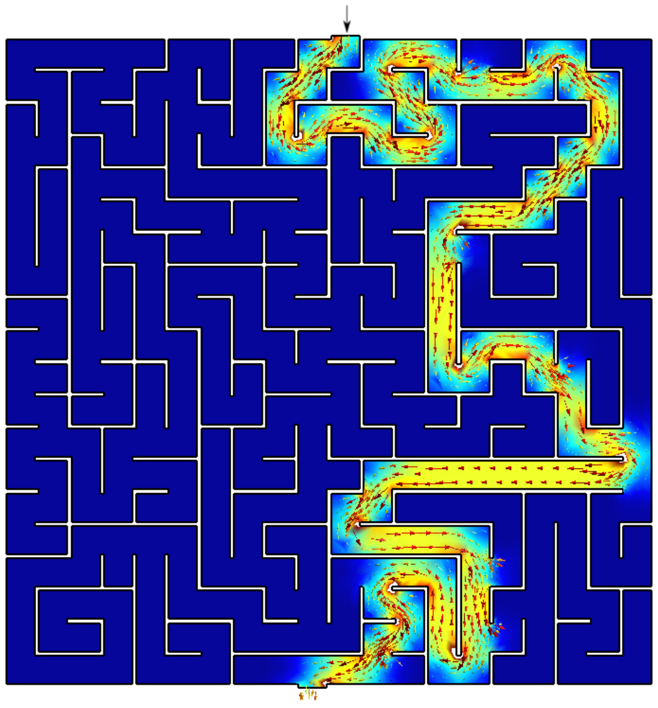


Figure 6: Solution to the maze found by FeenoX (and drawn by Gmsh)

```
PROBLEM laplace 2D
READ_MESH maze.msh

phi_0(x,y) = 0              # inital condition
end_time = 100              # some end time where we know we reached the steady-state
alpha = 1e-6                # factor of the time derivative to make it advance faster
BC start   phi=if(t<1,t,1)  # a ramp from zero to avoid discontinuities with the initial condition
BC end     phi=0            # homogeneous BC at the end (so we move from top to bottom)

SOLVE_PROBLEM
PRINT t

WRITE_MESH maze-tran-td.msh phi    sqrt(dphidx(x,y)^2+dphidy(x,y)^2) VECTOR -dphidx(x,y) -dphidy(x,y) 0
```

```
$ feenox maze-tran-td.fee
0
0.00433078
0.00949491
0.0170774
0.0268599
```

```
[...]
55.8631
64.0819
74.5784
87.2892
100
$ gmsh maze-tran-td-anim.geo
# all frames dumped, now run
ffmpeg -y -framerate 20 -f image2 -i maze-tran-td-%03d.png maze-tran-td.mp4
ffmpeg -y -framerate 20 -f image2 -i maze-tran-td-%03d.png maze-tran-td.gif
$ ffmpeg -y -framerate 20 -f image2 -i maze-tran-td-%03d.png maze-tran-td.mp4
[...]
$ ffmpeg -y -framerate 20 -f image2 -i maze-tran-td-%03d.png maze-tran-td.gif
[...]
```

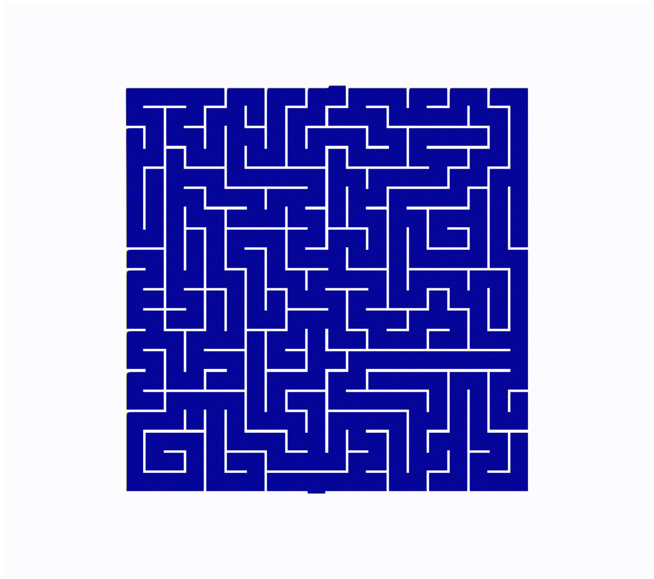


Figure 7: Transient top-bottom solution to the maze found by FeenoX (and drawn by Gmsh)

```
PROBLEM laplace 2D
READ_MESH maze.msh

phi_0(x,y) = 0
end_time = 100
alpha = 1e-6
BC end     phi=if(t<1,t,1)
BC start   phi=0

SOLVE_PROBLEM
PRINT t

WRITE_MESH maze-tran-bu.msh phi    sqrt(dphidx(x,y)^2+dphidy(x,y)^2) VECTOR -dphidx(x,y) -dphidy(x,y) 0
```

```
$ feenox maze-tran-bu.fee
0
```

```
0.00402961
0.00954806
0.0180156
0.0285787
[...]
65.3715
72.6894
81.8234
90.9117
100
$ gmsh maze-tran-bu-anim.geo
# all frames dumped, now run
ffmpeg -y -framerate 20 -f image2 -i maze-tran-bu-%03d.png maze-tran-bu.mp4
ffmpeg -y -framerate 20 -f image2 -i maze-tran-bu-%03d.png maze-tran-bu.gif
$ ffmpeg -y -framerate 20 -f image2 -i maze-tran-bu-%03d.png maze-tran-bu.mp4
[...]
$ ffmpeg -y -framerate 20 -f image2 -i maze-tran-bu-%03d.png maze-tran-bu.gif
[...]
```

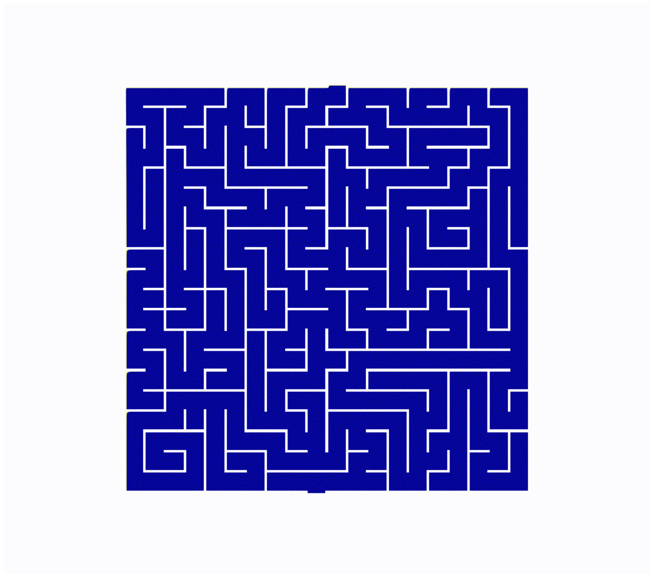


Figure 8: Transient bottom-up solution. The first attempt does not seem to be good.

```
# the fibonacci sequence as function
phi = (1+sqrt(5))/2
f(n) = (phi^n - (1-phi)^n)/sqrt(5)
PRINT_FUNCTION f MIN 1 MAX 20 STEP 1
```

```
$ feenox fibo_formula.fee | tee one
1       1
2       1
3       2
4       3
5       5
```

```
6       8
7       13
8       21
9       34
10      55
11      89
12      144
13      233
14      377
15      610
16      987
17      1597
18      2584
19      4181
20      6765
$
```

```
# the fibonacci sequence as a vector
VECTOR f SIZE 20

f[i]<1:2> = 1
f[i]<3:vecsize(f)> = f[i-2] + f[i-1]

PRINT_VECTOR i f
```

```
$ feenox fibo_vector.fee > two
$
```

```
static_steps = 20
#static_iterations = 1476  # limit of doubles

IF step_static=1|step_static=2
 f_n = 1
 f_nminus1 = 1
 f_nminus2 = 1
ELSE
 f_n = f_nminus1 + f_nminus2
 f_nminus2 = f_nminus1
 f_nminus1 = f_n
ENDIF

PRINT step_static f_n
```

```
$ feenox fibo_iterative.fee > three
$ diff one two
$ diff two three
$
```

```
#!/usr/local/bin/feenox

# read the function from stdin
FUNCTION f(t) FILE - INTERPOLATION steffen

# detect the domain range
```

```
a = vecmin(vec_f_t)
b = vecmax(vec_f_t)

# time step from arguments (or default 10 steps)
DEFAULT_ARGUMENT_VALUE 1 (b-a)/10
h = $1

# compute the derivative with a wrapper for gsl_deriv_central()
VAR t'
f'(t) = derivative(f(t'),t',t)

# write the result
PRINT_FUNCTION f' MIN a+0.5*h MAX b-0.5*h STEP h
```

```
$ chmod +x derivative.sh
$ feenox f.fee "sin(t)" 1 | ./derivative.fee 0.1 | tee f_prime.dat
0.05    0.998725
0.15    0.989041
0.25    0.968288
0.35    0.939643
0.45    0.900427
0.55    0.852504
0.65    0.796311
0.75    0.731216
0.85    0.66018
0.95    0.574296
$
```

```
PROBLEM elastic 3D
READ_MESH cantilever-$1-$2.msh   # in meters

E = 2.1e11         # Young modulus in Pascals
nu = 0.3           # Poisson's ratio

BC left   fixed
BC right  tz=-1e5  # traction in Pascals, negative z

SOLVE_PROBLEM

# z-displacement (components are u,v,w) at the tip vs. number of nodes
PRINT nodes %e w(500,0,0) "\# $1 $2"
```

```
$ ./cantilever.sh
102     -7.641572e-05   # tet4 1
495     -2.047389e-04   # tet4 2
1372    -3.149658e-04   # tet4 3
[...]
19737   -5.916234e-04   # hex27 8
24795   -5.916724e-04   # hex27 9
37191   -5.917163e-04   # hex27 10
$ pyxplot cantilever.ppl
```

```
PROBLEM modal 3D MODES 1  # only one mode needed
READ_MESH fork.msh  # in [m]
```

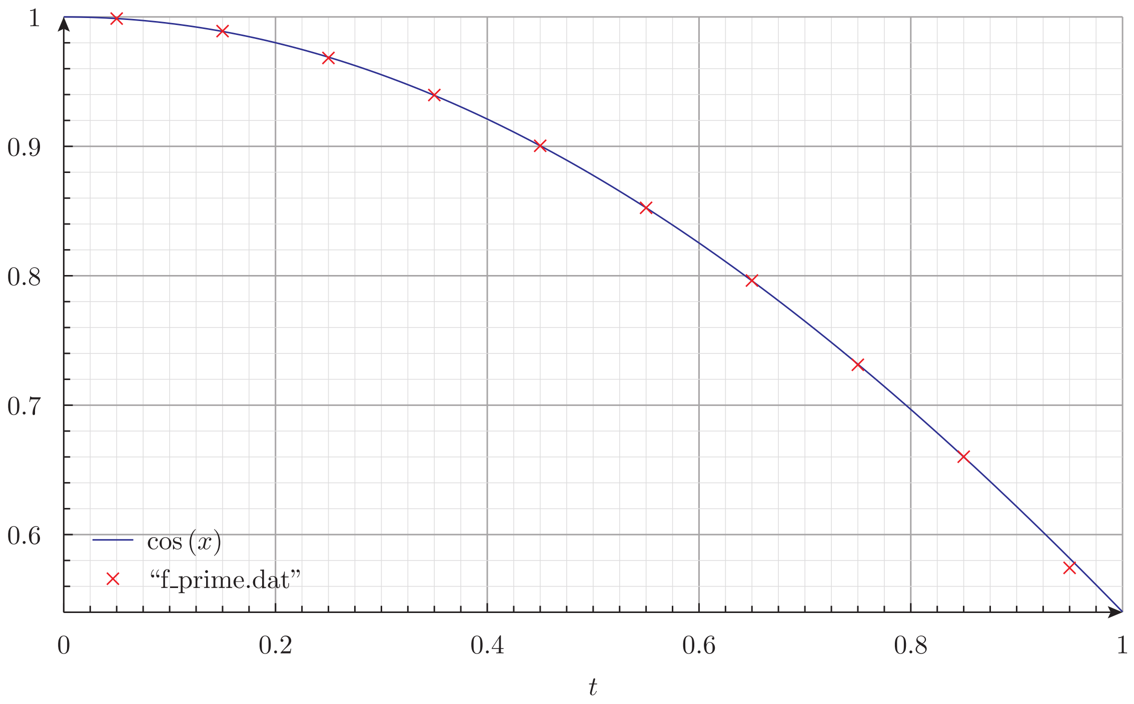


Figure 9: Numerical derivative as a UNIX filter and actual analytical result

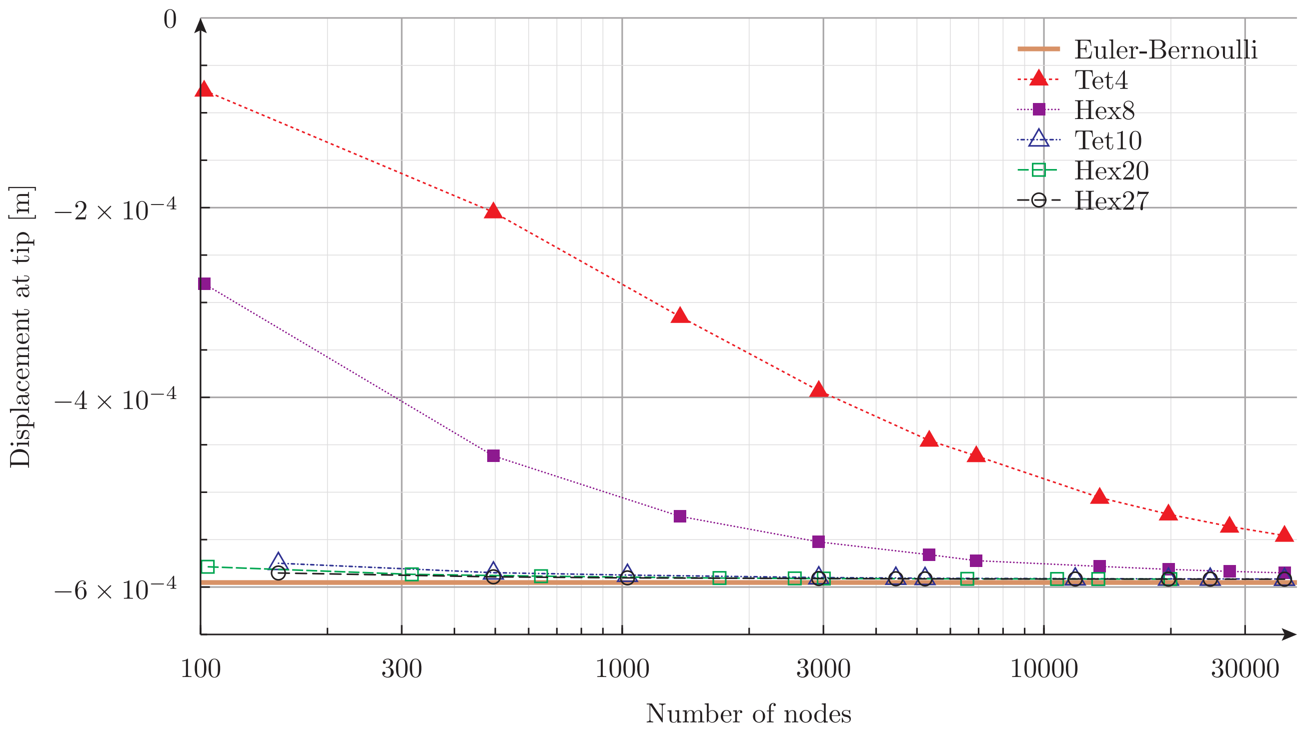


Figure 10: Displacement at the free tip of a cantilevered beam vs. number of nodes for different element types

```
E = 2.07e11         # in [Pa]
nu = 0.33
rho = 7829          # in [kg/m^2]

# no BCs! It is a free-free vibration problem
SOLVE_PROBLEM

# write back the fundamental frequency to stdout
PRINT f(1)
```

```
$ python fork.py > fork.dat
$ pyxplot fork.ppl
$
```

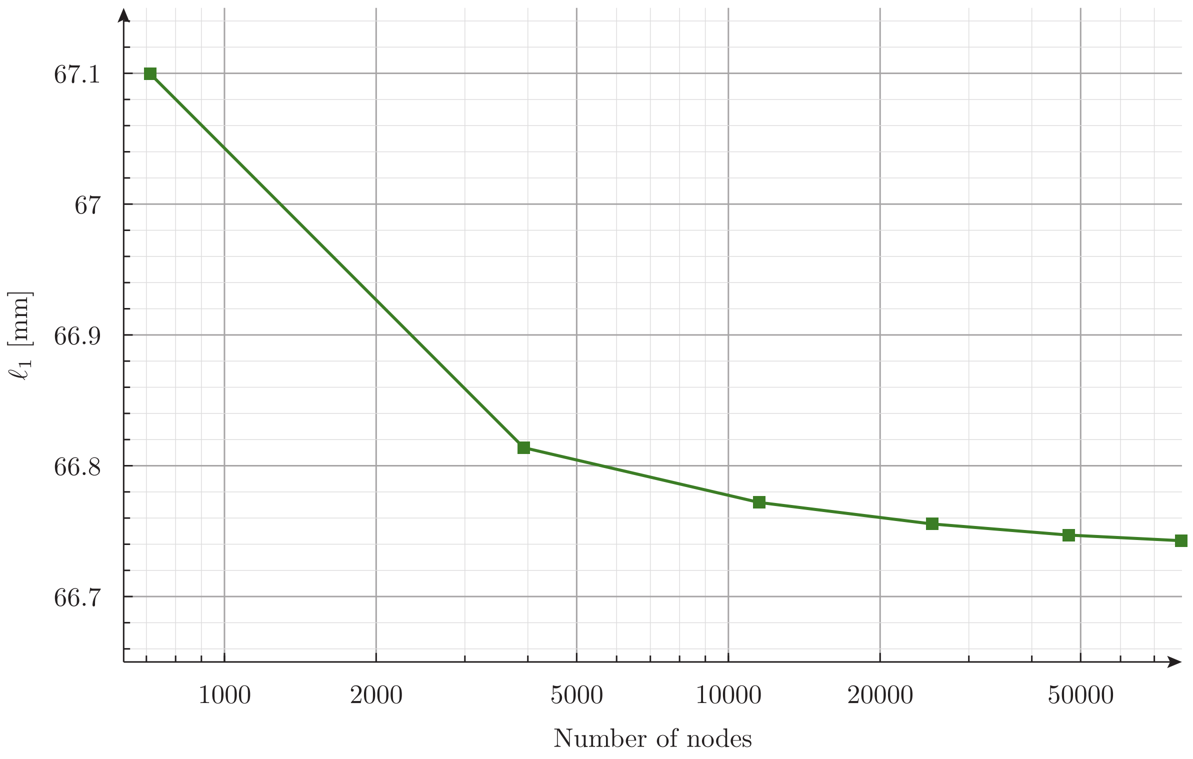


Figure 11: Estimated length $\ell_1$ needed to get 440 Hz for different mesh refinement levels $n$

```
#                       BENCHMARK PROBLEM
#
# Identification: 11-A2         Source Situation ID.11
# Date Submitted: June 1976     By: R. R. Lee (CE)
#                                   D. A. Menely (Ontario Hydro)
#                                   B. Micheelsen (Riso-Denmark)
#                                   D. R. Vondy (ORNL)
#                                   M. R. Wagner (KWU)
#                                   W. Werner (GRS-Munich)
#
# Date Accepted:  June 1977     By: H. L. Dodds, Jr. (U. of Tenn.)
#                                   M. V. Gregory (SRL)
```

```
#
# Descriptive Title: Two-dimensional LWR Problem,
#                    also 2D IAEA Benchmark Problem
#
# Reduction of Source Situation
#           1. Two-groupo diffusion theory
#           2. Two-dimensional (x,y)-geometry
#
PROBLEM neutron_diffusion 2D GROUPS 2
DEFAULT_ARGUMENT_VALUE 1 quarter   # either quarter or eigth
READ_MESH iaea-2dpwr-$1.msh

# define materials and cross sections according to the two-group constants
# each material corresponds to a physical entity in the geometry file
Bg2 = 0.8e-4  # axial geometric buckling in the z direction
MATERIAL fuel1 {
  D1=1.5    Sigma_a1=0.010+D1(x,y)*Bg2    Sigma_s1.2=0.02
  D2=0.4    Sigma_a2=0.080+D2(x,y)*Bg2    nuSigma_f2=0.135   }#eSigmaF_2 nuSigmaF_2(x,y) }

MATERIAL fuel2 {
  D1=1.5    Sigma_a1=0.010+D1(x,y)*Bg2    Sigma_s1.2=0.02
  D2=0.4    Sigma_a2=0.085+D2(x,y)*Bg2    nuSigma_f2=0.135   }#eSigmaF_2 nuSigmaF_2(x,y) }

MATERIAL fuel2rod {
  D1=1.5    Sigma_a1=0.010+D1(x,y)*Bg2    Sigma_s1.2=0.02
  D2=0.4    Sigma_a2=0.130+D2(x,y)*Bg2    nuSigma_f2=0.135   }#eSigmaF_2 nuSigmaF_2(x,y) }

MATERIAL reflector {
  D1=2.0    Sigma_a1=0.000+D1(x,y)*Bg2    Sigma_s1.2=0.04
  D2=0.3    Sigma_a2=0.010+D2(x,y)*Bg2 }


# define boundary conditions as requested by the problem
BC external vacuum=0.4692  # "external" is the name of the entity in the .geo
BC mirror   mirror         # the first mirror is the name, the second is the BC type

# # set the power setpoint equal to the volume of the core
# # (and set eSigmaF_2 = nuSigmaF_2 as above)
# power = 17700

SOLVE_PROBLEM   # solve!
PRINT %.5f "keff = " keff
WRITE_MESH iaea-2dpwr-$1.vtk phi1 phi2
```

```
$ gmsh -2 iaea-2dpwr-quarter.geo
$ [...]
$ gmsh -2 iaea-2dpwr-eighth.geo
$ [...]
$ feenox iaea-2dpwr.fee quarter
keff =  1.02986
$ feenox iaea-2dpwr.fee eighth
$keff =  1.02975
$
```

```
PROBLEM neutron_diffusion DIMENSIONS 3
READ_MESH cubesphere-$1.msh DIMENSIONS 3
```

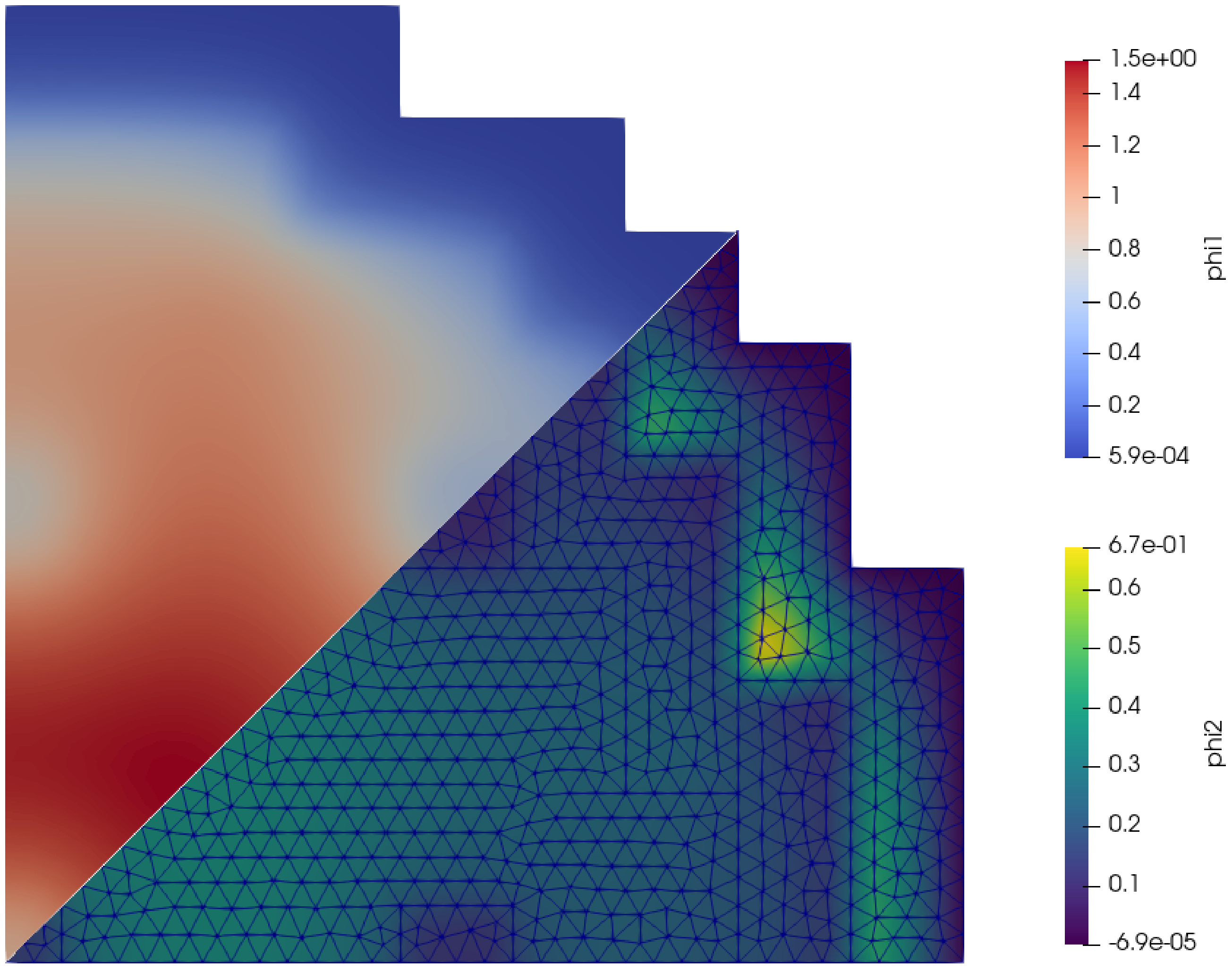


Figure 12: Fast and thermal flux for the 2D IAEA PWR benchmark (2021)

```
# MATERIAL fuel
D1 = 1.03453E+00
Sigma_a1 = 5.59352E-03
nuSigma_f1 = 6.68462E-03
Sigma_s1.1 = 3.94389E-01

PHYSICAL_GROUP fuel DIM 3
BC internal    mirror
BC external    vacuum

SOLVE_PROBLEM

PRINT HEADER $1 keff 1e5*(keff-1)/keff fuel_volume
```

```
$ python cubesphere.py | tee cubesphere.dat
0       1.05626 5326.13 1e+06
5       1.05638 5337.54 999980
10      1.05675 5370.58 999980
15      1.05734 5423.19 999992
20      1.05812 5492.93 999995
25      1.05906 5576.95 999995
30      1.06013 5672.15 999996
35      1.06129 5775.31 999997
40      1.06251 5883.41 999998
45      1.06376 5993.39 999998
50      1.06499 6102.55 999998
55      1.06619 6208.37 999998
60      1.06733 6308.65 999998
65      1.06839 6401.41 999999
70      1.06935 6485.03 999998
75      1.07018 6557.96 999998
80      1.07088 6618.95 999998
85      1.07143 6666.98 999999
90      1.07183 6701.24 999999
95      1.07206 6721.33 999998
100     1.07213 6727.64 999999
$
```

```
#  FeenoX of course can solve both cases, but there are many other neutronic tools out there that can ↩
     handle ony structured grids.
PROBLEM neutron_diffusion 1D
DEFAULT_ARGUMENT_VALUE 1 nonuniform
READ_MESH two-zone-slab-$1.msh

# this ab.geo is created from the driving shell script
INCLUDE ab.geo

# pure material A from x=0 to x=a
D1_A = 0.5
Sigma_a1_A = 0.014
nuSigma_f1_A = 0.010

# pure material B from x=a to x=b
D1_B = 1.2
Sigma_a1_B = 0.010
```

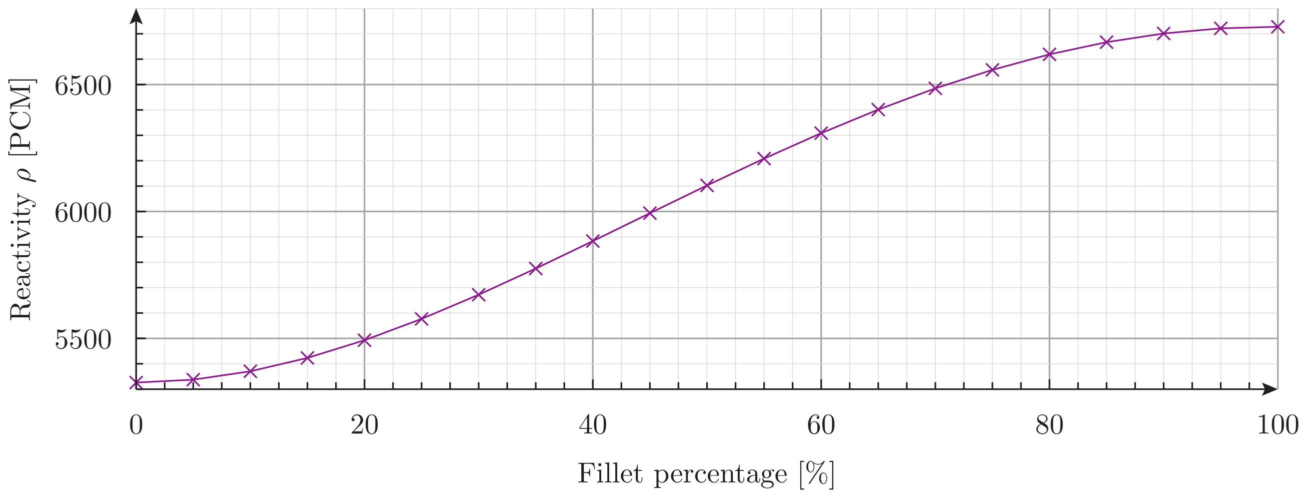


Figure 13: Static reactivity vs. percentage of sphericity

```
nuSigma_f1_B = 0.014

# meta-material (only used for uniform grid to illustrate XS dilution)
a_left  = floor(a/lc)*lc;
xi = (a - a_left)/lc
Sigma_tr_A = 1/(3*D1_A)
Sigma_tr_B = 1/(3*D1_B)
Sigma_tr_AB = xi*Sigma_tr_A + (1-xi)*Sigma_tr_B
D1_AB = 1/(3*Sigma_tr_AB)
Sigma_a1_AB = xi * Sigma_a1_A + (1-xi)*Sigma_a1_B
nuSigma_f1_AB = xi * nuSigma_f1_A + (1-xi)*nuSigma_f1_B

BC left  null
BC right null

SOLVE_PROBLEM


# compute the analytical keff
F1(k) = sqrt(D1_A*(Sigma_a1_A-nuSigma_f1_A/k)) * tan(sqrt((1/D1_B)*(nuSigma_f1_B/k-Sigma_a1_B))*(a-b))
F2(k) = sqrt(D1_B*(nuSigma_f1_B/k-Sigma_a1_B)) * tanh(sqrt((1/D1_A)*(Sigma_a1_A-nuSigma_f1_A/k))*b)
k = root(F1(k)-F2(k), k, 1, 1.2)

# # and the fluxes (not needed here but for reference)
# B_A = sqrt((Sigma_a1_A - nuSigma_f1_A/k)/D1_A)
# fluxA(x) = sinh(B_A*x)
#
# B_B = sqrt((nuSigma_f1_B/k - Sigma_a1_B)/D1_B)
# fluxB(x)= sinh(B_A*b)/sin(B_B*(a-b)) * sin(B_B*(a-x))
#
# # normalization factor
# f = a/(integral(fluxA(x), x, 0, b) + integral(fluxB(x), x, b, a))
# flux(x) := f * if(x < b, fluxA(x), fluxB(x))

PRINT a keff k keff-k b n lc nodes

# PRINT_FUNCTION flux MIN 0 MAX a STEP a/1000 FILE_PATH two-zone-analytical.dat
# PRINT_FUNCTION phi1 phi1(x)-flux(x)          FILE_PATH two-zone-numerical.dat
```

```
$ ./two-zone-slab.sh 10
[...]
$ ./two-zone-slab.sh 20
[...]
$ pyxplot two-zone-slab.ppl
$
```

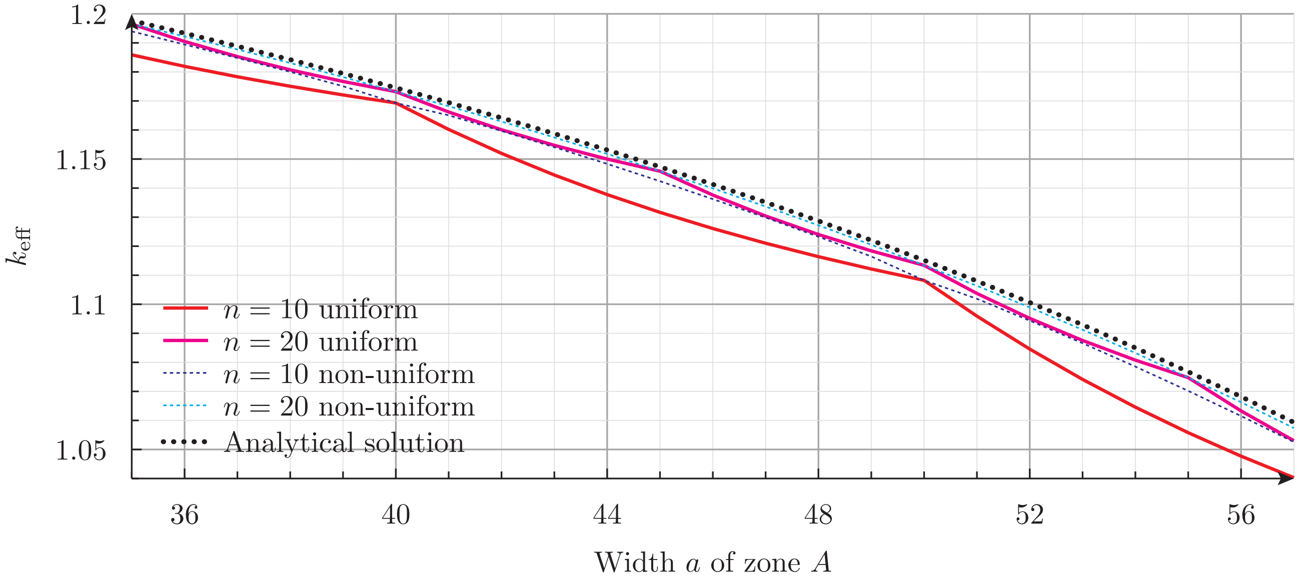


Figure 14: $k_{\text{eff}}$ vs. $a$

> To illustrate the point of this example, think about the following 2D case:

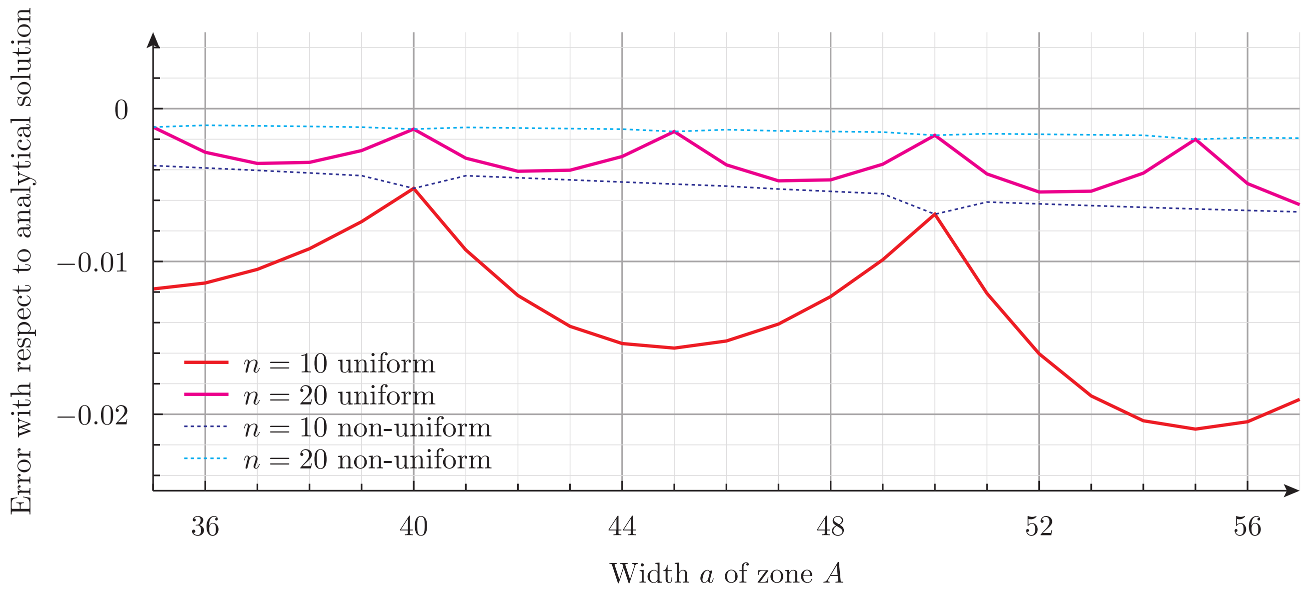
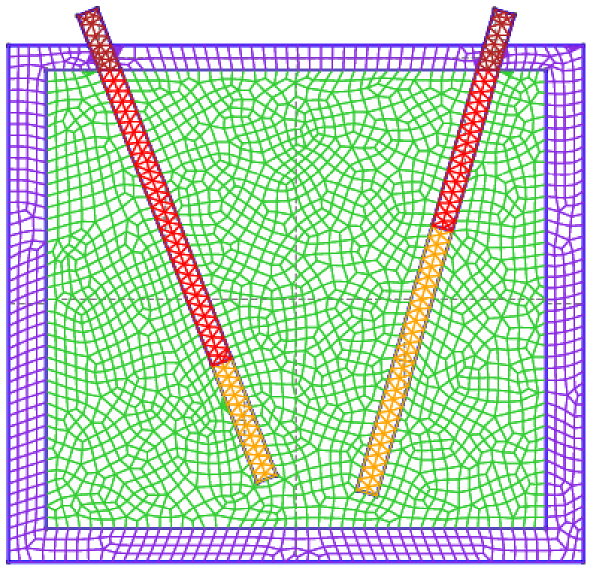


Figure 15: Error vs. $a$



1. How would you solve something like this with a neutronic tool that only allowed structured grids?
2. Even if the two control rods were not slanted, as long as they were not inserted up to the same height there would be XS dilution & semaring when using a structured grid (even if the tool allows non-uniform cells in each direction).
3. Consider RMS's quotation above: the best way to solve a problem (i.e. XS dilution) is to avoid it in the first place (i.e. to use a tool able to handle unstructured grids).

```
PROBLEM thermal 3D
```

```
READ_MESH parallelepiped-coarse.msh

k = 1     # unitary non-dimensional thermal conductivity

# boundary conditions
BC left    q=+2
BC right   q=-2
BC front   q=+3
BC back    q=-3
BC bottom  q=+4
BC top     q=-4
BC A       T=0

SOLVE_PROBLEM
WRITE_MESH parallelepiped-temperature.msh T

# compute the L-2 norm of the error in the displacement field
Te(x,y,z) = 40 - 2*x - 3*y - 4*z   # analytical solution, "e" means exact
INTEGRATE (T(x,y,z)-Te(x,y,z))^2 RESULT num
PHYSICAL_GROUP bulk DIM 3  # this is just to compute the volume
PRINT num/bulk_volume
```

```
$ gmsh -3 parallelepiped.geo -order 1 -clscale 2 -o parallelepiped-coarse.msh
[...]
Info    : 117 nodes 567 elements
Info    : Writing 'parallelepiped-coarse.msh'...
Info    : Done writing 'parallelepiped-coarse.msh'
Info    : Stopped on Fri Dec 10 10:32:30 2021 (From start: Wall 0.0386516s, CPU 0.183052s)
$ feenox parallelepiped-thermal.fee
6.18981e-12
$
```

```
PROBLEM mechanical 3D

# this is where we solve the mechanical problem
READ_MESH parallelepiped.msh MAIN

# this is where we read the temperature from
READ_MESH parallelepiped-temperature.msh DIM 3 READ_FUNCTION T

# mechanical properties
E(x,y,z) = 1000/(800-T(x,y,z))    # young's modulus
nu = 0.3                          # poisson's ratio

# boundary conditions
BC O fixed
BC B u=0 w=0
BC C u=0

# here "load" is a fantasy name applied to both "left" and "right"
BC load tension=1 PHYSICAL_GROUP left PHYSICAL_GROUP right

SOLVE_PROBLEM
WRITE_MESH parallelepiped-mechanical.vtk T VECTOR u v w

# analytical solutions
h = 10
```

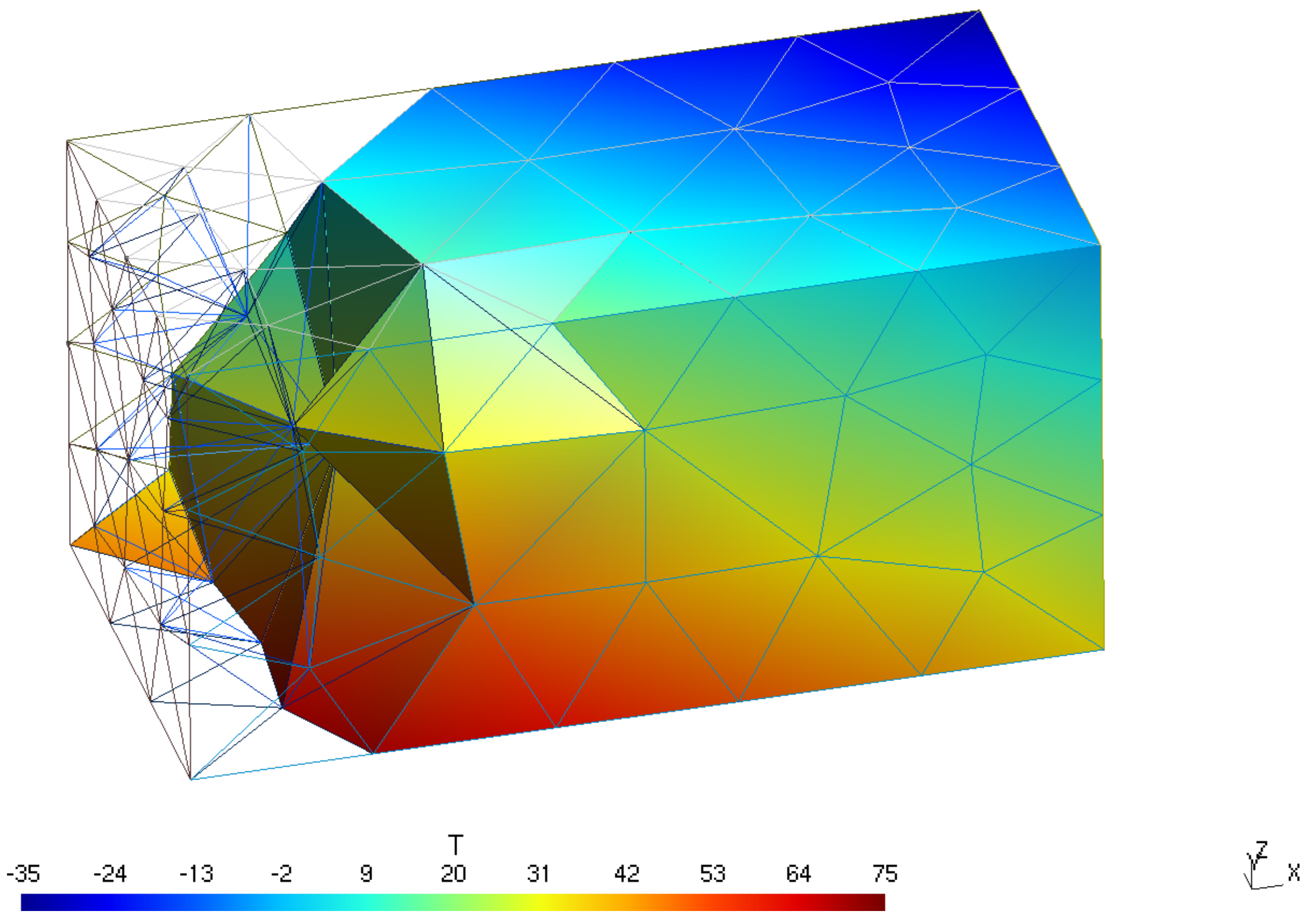


Figure 16: Temperature distribution over the coarse mesh in Gmsh (yes, it is a rainbow pallete)

```
A = 0.002
B = 0.003
C = 0.004
D = 0.76

# the "e" means exact
ue(x,y,z) := A/2*(x^2 + nu*(y^2+z^2)) + B*x*y + C*x*z + D*x - nu*A*h/4*(y+z)
ve(x,y,z) := -nu*(A*x*y + B/2*(y^2-z^2+x^2/nu) + C*y*z + D*y -A*h/4*x - C*h/4*z)
we(x,y,z) := -nu*(A*x*z + B*y*z + C/2*(z^2-y^2+x^2/nu) + D*z + C*h/4*y - A*h/4*x)

# compute the L-2 norm of the error in the displacement field
INTEGRATE (u(x,y,z)-ue(x,y,z))^2+(v(x,y,z)-ve(x,y,z))^2+(w(x,y,z)-we(x,y,z))^2 RESULT num MESH ↩
    parallelepiped.msh
INTEGRATE 1 RESULT den MESH parallelepiped.msh
PRINT num/den
```

```
$ gmsh -3 parallelepiped.geo -order 2
[...]
Info    : 2564 nodes 2162 elements
Info    : Writing 'parallelepiped.msh'...
Info    : Done writing 'parallelepiped.msh'
Info    : Stopped on Fri Dec 10 10:39:27 2021 (From start: Wall 0.165707s, CPU 0.258751s)
$ feenox parallelepiped-mechanical.fee
0.000345839
$
```

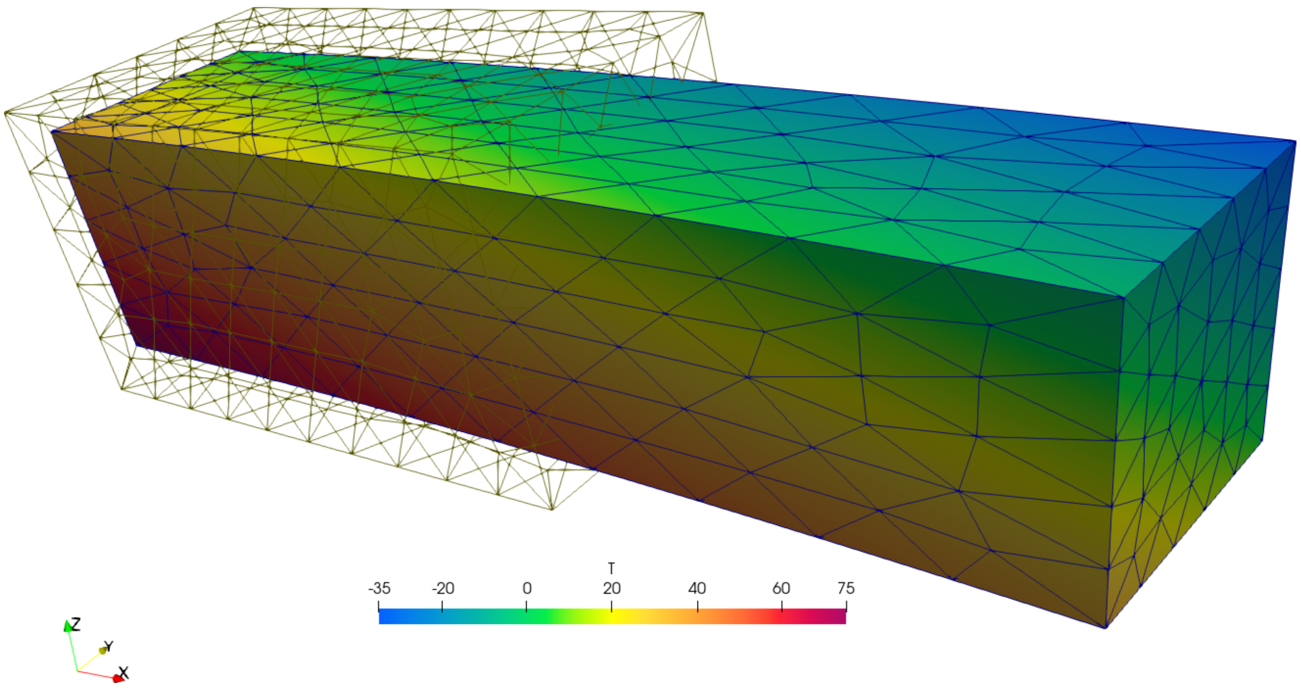


Figure 17: Displacements and temperature distribution over the fine mesh in Paraview (yes, still a rainbow pallete)

```
PROBLEM thermal 3D
READ_MESH cylinder.msh
```

```
end_time = 2  # final time [ non-dimensional units ]
# the time step is automatically computed

# initial condition (if not given, stead-state is computed)
T_0(x,y,z) = x

# dimensionless uniform and constant material properties
k = 1
kappa = 1

# BCs
BC hot   T=sin(2*pi*t)*sin(2*pi*y)
BC cool  h=0.1  Tref=1

SOLVE_PROBLEM

# print the temperature at the center of the base vs time
PRINT %e t T(0,0,0) T(0.5,0,0) T(1,0,0)

WRITE_MESH temp-cylinder.msh T

IF done
 PRINT "\# open temp-anim-cylinder.geo in Gmsh to create a quick rough video"
 PRINT "\# run  temp-anim-cylinder.py  to get a nicer and smoother video"
ENDIF
```

```
$ gmsh -3 cylinder.geo
[...]
Info    : Done optimizing mesh (Wall 0.624941s, CPU 0.624932s)
Info    : 1986 nodes 10705 elements
Info    : Writing 'cylinder.msh'...
Info    : Done writing 'cylinder.msh'
Info    : Stopped on Fri Dec 24 10:35:32 2021 (From start: Wall 0.800542s, CPU 0.896698s)
$ feenox temp-cylinder-tran.fee
0.000000e+00    0.000000e+00    5.000000e-01    1.000000e+00
1.451938e-04    4.406425e-07    5.000094e-01    9.960851e-01
3.016938e-04    9.155974e-07    5.000171e-01    9.921274e-01
5.566768e-04    1.689432e-06    5.000251e-01    9.862244e-01
8.565589e-04    2.599523e-06    5.000292e-01    9.800113e-01
1.245867e-03    3.780993e-06    5.000280e-01    9.728705e-01
1.780756e-03    5.404230e-06    5.000176e-01    9.643259e-01
2.492280e-03    7.563410e-06    4.999932e-01    9.545723e-01
3.428621e-03    1.040457e-05    4.999538e-01    9.436480e-01
[...]
1.978669e+00    -6.454358e-05   1.500891e-01    2.286112e-01
1.989334e+00    -3.234439e-05   1.500723e-01    2.285660e-01
2.000000e+00    1.001730e-14    1.500572e-01    2.285223e-01
# open temp-anim-cylinder.geo in Gmsh to create a quick rough video
# run  temp-anim-cylinder.py  to get a nicer and smoother video
$ python3 temp-anim-cylinder.py
Info    : Reading 'temp-cylinder.msh'...
Info    : 1986 nodes
Info    : 10612 elements
Info    : Done reading 'temp-cylinder.msh'
0 1 0.0
```

```
0.01 12 0.8208905327853042
0.02 15 0.8187351216040447
0.03 17 0.7902629708599855
[...]
Info    : Writing 'temp-cylinder-smooth-198.png'...
Info    : Done writing 'temp-cylinder-smooth-198.png'
199
Info    : Writing 'temp-cylinder-smooth-199.png'...
Info    : Done writing 'temp-cylinder-smooth-199.png'
all frames dumped, now run
ffmpeg -framerate 20 -f image2 -i temp-cylinder-smooth-%03d.png temp-cylinder-smooth.mp4
to get a video
$ ffmpeg -y -f image2 -i temp-cylinder-smooth-%03d.png  -framerate 20 -pix_fmt yuv420p -c:v libx264 -filter: ↩
    v crop='floor(in_w/2)*2:floor(in_h/2)*2'  temp-cylinder-smooth.mp4
[...]
$
```

```
#
DEFAULT_ARGUMENT_VALUE 1 hex       # mesh, either hex or unstruct
DEFAULT_ARGUMENT_VALUE 2 copper    # material, either copper or aluminum

l = 0.5*303e-3   # cantilever wire length [ m ]
d = 1.948e-3     # wire diameter [ m ]


# material properties for copper
m_copper = 0.5*8.02e-3  # total mass (half the measured because of the experimental disposition) [ kg ]
E_copper = 2*66.2e9     # [ Pa ] Young modulus (twice because the factor-two error)

# material properties for aluminum
m_aluminum = 0.5*2.67e-3
E_aluminum = 2*40.2e9

# 'problems properties
E = E_$2                      # [ MPa ]
rho = m_$2/(pi*(0.5*d)^2*l)   # [ kg / m^3 ] density = mass (measured) / volume
nu = 0                        # 'Poissons ratio (does not appear in Euler-Bernoulli)

# analytical solution
VECTOR kl[5]
VECTOR f_euler[5]

# first compute the first five roots ok cosh(kl)*cos(kl)+1
kl[i] = root(cosh(t)*cos(t)+1, t, 3*i-2,3*i+1)

# then compute the frequencies according to Euler-Bernoulli
# note that we need to use SI inside the square root
A = pi * (d/2)^2
I = pi/4 * (d/2)^4
f_euler[i] = 1/(2*pi) * kl(i)^2 * sqrt((E * I)/(rho * A * l^4))

# now compute the modes numerically with FEM
# note that each mode is duplicated as it is degenerated
READ_MESH wire-$1.msh DIMENSIONS 3
PROBLEM modal MODES 10
BC fixed fixed
SOLVE_PROBLEM
```

```
# github-formatted markdown table
# compare the frequencies
PRINT "  \$n\$   |    FEM [Hz]   |   Euler [Hz]  |  Relative difference [%]"
PRINT   ":------:|:-------------:|:-------------:|:--------------:"
PRINT_VECTOR SEP "\t|\t" i  %.4g f(2*i-1) f_euler   %.2f 100*(f_euler(i)-f(2*i-1))/f_euler(i)
PRINT
PRINT ": $2 wire over $1 mesh"

# commonmark table
PRINT
PRINT "  \$n\$   |          \$L\$          |       \$\\Gamma\$        |      \$\\mu\$    |       \$M\$"
PRINT   ":------:+:--------------------:+:--------------------:+:-------------:+:--------------:"
PRINT_VECTOR SEP "\t|\t" i "%+.1e" L Gamma "%.4f" mu Mu
PRINT
PRINT ": $2 wire over $1 mesh, participation and excitation factors \$L\$ and \$\\Gamma\$, effective  ↩
     per-mode and cummulative mass fractions \$\\mu\$ and \$M\$"

# write the modes into a vtk file
WRITE_MESH wire-$1-$2.vtk \
 VECTOR u1 v1 w1 VECTOR u2 v2 w2 VECTOR u3 v3 w3 \
 VECTOR u4 v4 w4 VECTOR u5 v5 w5 VECTOR u6 v6 w6 \
 VECTOR u7 v7 w7 VECTOR u8 v8 w8 VECTOR u9 v9 w9 VECTOR u10 v10 w10

# and into a msh file
WRITE_MESH wire-$1-$2.msh {
 u1 v1 w1
 u2 v2 w2
 u3 v3 w3
 u4 v4 w4
 u5 v5 w5
 u6 v6 w6
 u7 v7 w7
 u8 v8 w8
 u9 v9 w9
 u10 v10 w10
}
```

```
$ gmsh -3 wire-hex.geo
[...]
Info    : Done meshing order 2 (Wall 0.0169025s, CPU 0.016804s)
Info    : 8398 nodes 4676 elements
Info    : Writing 'wire-hex.msh'...
Info    : Done writing 'wire-hex.msh'
Info    : Stopped on Fri Dec 24 17:07:19 2021 (From start: Wall 0.0464517s, CPU 0.133498s)
$ gmsh -3 wire-tet.geo
[...]
Info    : Done optimizing mesh (Wall 0.0229018s, CPU 0.022892s)
Info    : 16579 nodes 13610 elements
Info    : Writing 'wire-tet.msh'...
Info    : Done writing 'wire-tet.msh'
Info    : Stopped on Fri Dec 24 17:07:59 2021 (From start: Wall 2.5798s, CPU 2.64745s)
$ feenox wire.fee
  $n$   |    FEM [Hz]   |   Euler [Hz]  |  Relative difference [%]
:------:|:-------------:|:-------------:|:--------------:
1       |       45.84   |       45.84   |       0.02
2       |       287.1   |       287.3   |       0.06
```

```
3       |       803.4   |       804.5   |       0.13
4       |       1573    |       1576    |       0.24
5       |       2596    |       2606    |       0.38

: copper wire over hex mesh

  $n$   |          $L$          |       $\Gamma$        |      $\mu$    |       $M$
:------:+:--------------------:+:--------------------:+:------------:+:-------------:
1       |       +1.3e-03        |       +4.2e-01        |       0.1371  |       0.1371
2       |       -1.8e-03        |       -5.9e-01        |       0.2716  |       0.4087
3       |       +9.1e-05        |       +1.7e-02        |       0.0004  |       0.4091
4       |       -1.7e-03        |       -3.0e-01        |       0.1252  |       0.5343
5       |       -3.3e-05        |       -5.9e-03        |       0.0000  |       0.5343
6       |       -9.9e-04        |       -1.8e-01        |       0.0431  |       0.5775
7       |       +7.3e-04        |       +1.2e-01        |       0.0221  |       0.5995
8       |       +4.5e-06        |       +7.5e-04        |       0.0000  |       0.5995
9       |       +5.4e-04        |       +9.9e-02        |       0.0134  |       0.6129
10      |       +2.7e-05        |       +4.9e-03        |       0.0000  |       0.6129

: copper wire over hex mesh, participation and excitation factors $L$ and $\Gamma$, effective per-mode and
    cummulative mass fractions $\mu$ and $M$
$ feenox wire.fee hex copper   | pandoc -o wire-hex-copper.md
$ feenox wire.fee tet copper   | pandoc -o wire-tet-copper.md
$ feenox wire.fee hex aluminum | pandoc -o wire-hex-aluminum.md
$ feenox wire.fee tet aluminum | pandoc -o wire-tet-aluminum.md
```

Table 1: copper wire over hex mesh

| $n$ | FEM [Hz] | Euler [Hz] | Relative difference [%] |
|---|---|---|---|
| 1 | 45.84 | 45.84 | 0.02 |
| 2 | 287.1 | 287.3 | 0.06 |
| 3 | 803.4 | 804.5 | 0.13 |
| 4 | 1573 | 1576 | 0.24 |
| 5 | 2596 | 2606 | 0.38 |

Table 2: copper wire over hex mesh, participation and excitation factors $L$ and $\Gamma$, effective per-mode and cummulative mass fractions $\mu$ and $M$

| $n$ | $L$ | $\Gamma$ | $\mu$ | $M$ |
|---|---|---|---|---|
| 1 | -1.8e-03 | -5.9e-01 | 0.2713 | 0.2713 |
| 2 | +1.3e-03 | +4.2e-01 | 0.1374 | 0.4087 |
| 3 | +9.7e-05 | +1.8e-02 | 0.0004 | 0.4091 |
| 4 | -1.6e-03 | -3.1e-01 | 0.1251 | 0.5343 |
| 5 | -3.5e-05 | -6.3e-03 | 0.0001 | 0.5343 |
| 6 | -9.9e-04 | -1.8e-01 | 0.0431 | 0.5774 |
| 7 | +7.2e-04 | +1.2e-01 | 0.0221 | 0.5995 |

| $n$ | $L$ | $\Gamma$ | $\mu$ | $M$ |
|---|---|---|---|---|
| 8 | -8.6e-06 | -1.5e-03 | 0.0000 | 0.5995 |
| 9 | -2.6e-05 | -4.7e-03 | 0.0000 | 0.5996 |
| 10 | +5.4e-04 | +9.9e-02 | 0.0134 | 0.6130 |

Table 3: copper wire over tet mesh

| $n$ | FEM [Hz] | Euler [Hz] | Relative difference [%] |
|---|---|---|---|
| 1 | 45.84 | 45.84 | 0.00 |
| 2 | 287.2 | 287.3 | 0.05 |
| 3 | 803.4 | 804.5 | 0.13 |
| 4 | 1573 | 1576 | 0.24 |
| 5 | 2596 | 2606 | 0.38 |

Table 4: copper wire over tet mesh, participation and excitation factors $L$ and $\Gamma$, effective per-mode and cummulative mass fractions $\mu$ and $M$

| $n$ | $L$ | $\Gamma$ | $\mu$ | $M$ |
|---|---|---|---|---|
| 1 | -1.9e-03 | -6.1e-01 | 0.2925 | 0.2925 |
| 2 | +1.2e-03 | +3.8e-01 | 0.1163 | 0.4087 |
| 3 | -1.0e-03 | -3.3e-01 | 0.0861 | 0.4948 |
| 4 | +7.0e-04 | +2.3e-01 | 0.0395 | 0.5343 |
| 5 | -6.0e-04 | -1.9e-01 | 0.0292 | 0.5634 |
| 6 | +4.2e-04 | +1.3e-01 | 0.0140 | 0.5774 |
| 7 | -4.0e-04 | -1.3e-01 | 0.0133 | 0.5908 |
| 8 | +3.3e-04 | +1.1e-01 | 0.0087 | 0.5995 |
| 9 | +3.5e-04 | +1.1e-01 | 0.0096 | 0.6091 |
| 10 | -2.2e-04 | -6.9e-02 | 0.0038 | 0.6129 |

Table 5: aluminum wire over hex mesh

| $n$ | FEM [Hz] | Euler [Hz] | Relative difference [%] |
|---|---|---|---|
| 1 | 61.91 | 61.92 | 0.02 |
| 2 | 387.8 | 388 | 0.06 |
| 3 | 1085 | 1086 | 0.13 |
| 4 | 2124 | 2129 | 0.24 |
| 5 | 3506 | 3519 | 0.38 |

Table 6: aluminum wire over hex mesh, participation and excitation factors $L$ and $\Gamma$, effective per-mode and cummulative mass fractions $\mu$ and $M$

| $n$ | $L$ | $\Gamma$ | $\mu$ | $M$ |
|---|---|---|---|---|
| 1 | -6.9e-04 | -6.2e-01 | 0.3211 | 0.3211 |
| 2 | +3.6e-04 | +3.3e-01 | 0.0876 | 0.4087 |
| 3 | +4.2e-05 | +2.4e-02 | 0.0008 | 0.4095 |
| 4 | -5.4e-04 | -3.1e-01 | 0.1248 | 0.5343 |
| 5 | +3.7e-05 | +2.3e-02 | 0.0006 | 0.5349 |
| 6 | -3.0e-04 | -1.9e-01 | 0.0425 | 0.5774 |
| 7 | +2.4e-04 | +1.2e-01 | 0.0221 | 0.5995 |
| 8 | -3.2e-06 | -1.6e-03 | 0.0000 | 0.5995 |
| 9 | +1.8e-04 | +9.8e-02 | 0.0132 | 0.6127 |
| 10 | -9.5e-06 | -5.2e-03 | 0.0000 | 0.6128 |

Table 7: aluminum wire over tet mesh

| $n$ | FEM [Hz] | Euler [Hz] | Relative difference [%] |
|---|---|---|---|
| 1 | 61.91 | 61.92 | 0.00 |
| 2 | 387.8 | 388 | 0.05 |
| 3 | 1085 | 1086 | 0.13 |
| 4 | 2124 | 2129 | 0.24 |
| 5 | 3506 | 3519 | 0.38 |

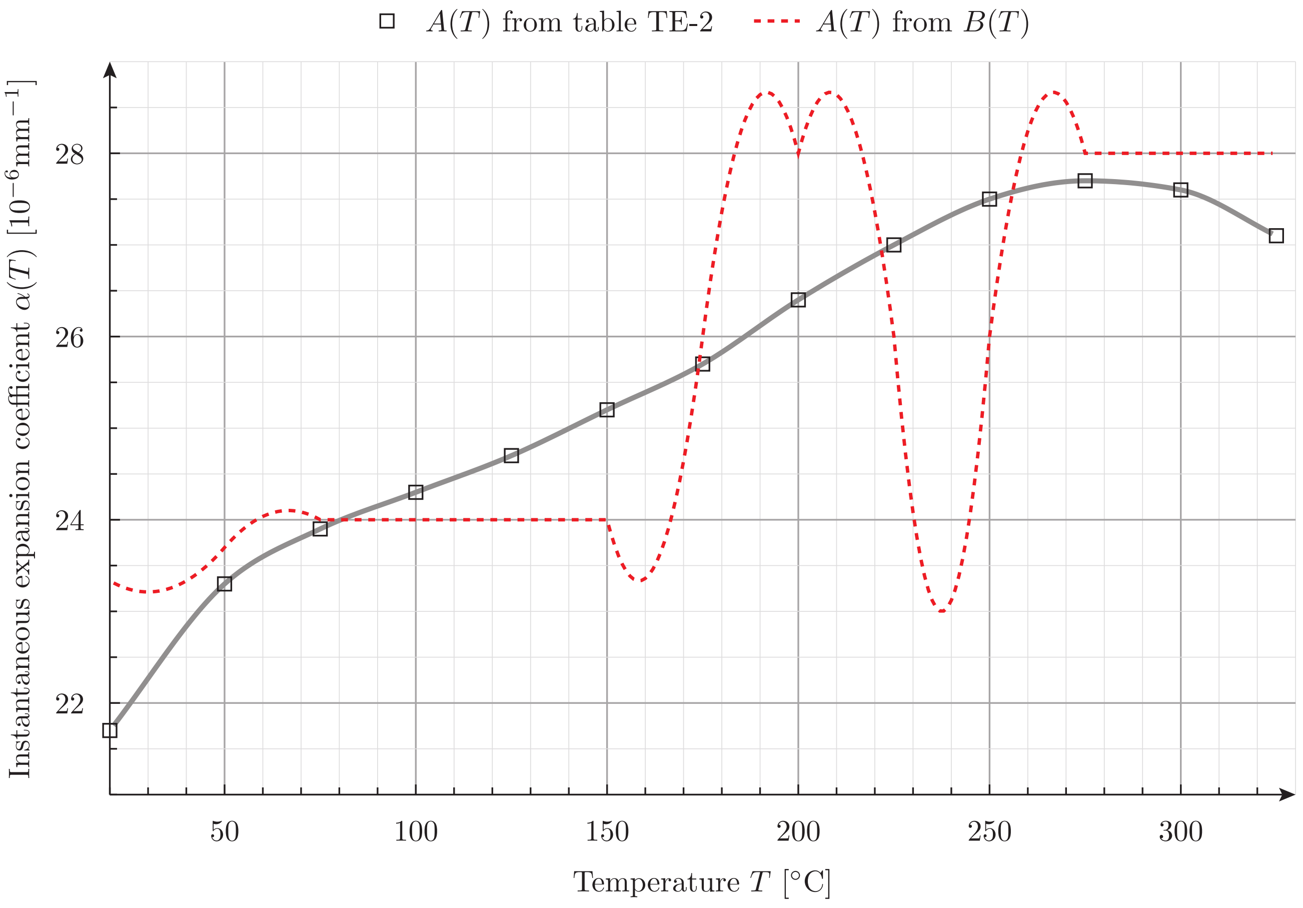
Table 8: aluminum wire over tet mesh, participation and excitation factors $L$ and $\Gamma$, effective per-mode and cummulative mass fractions $\mu$ and $M$

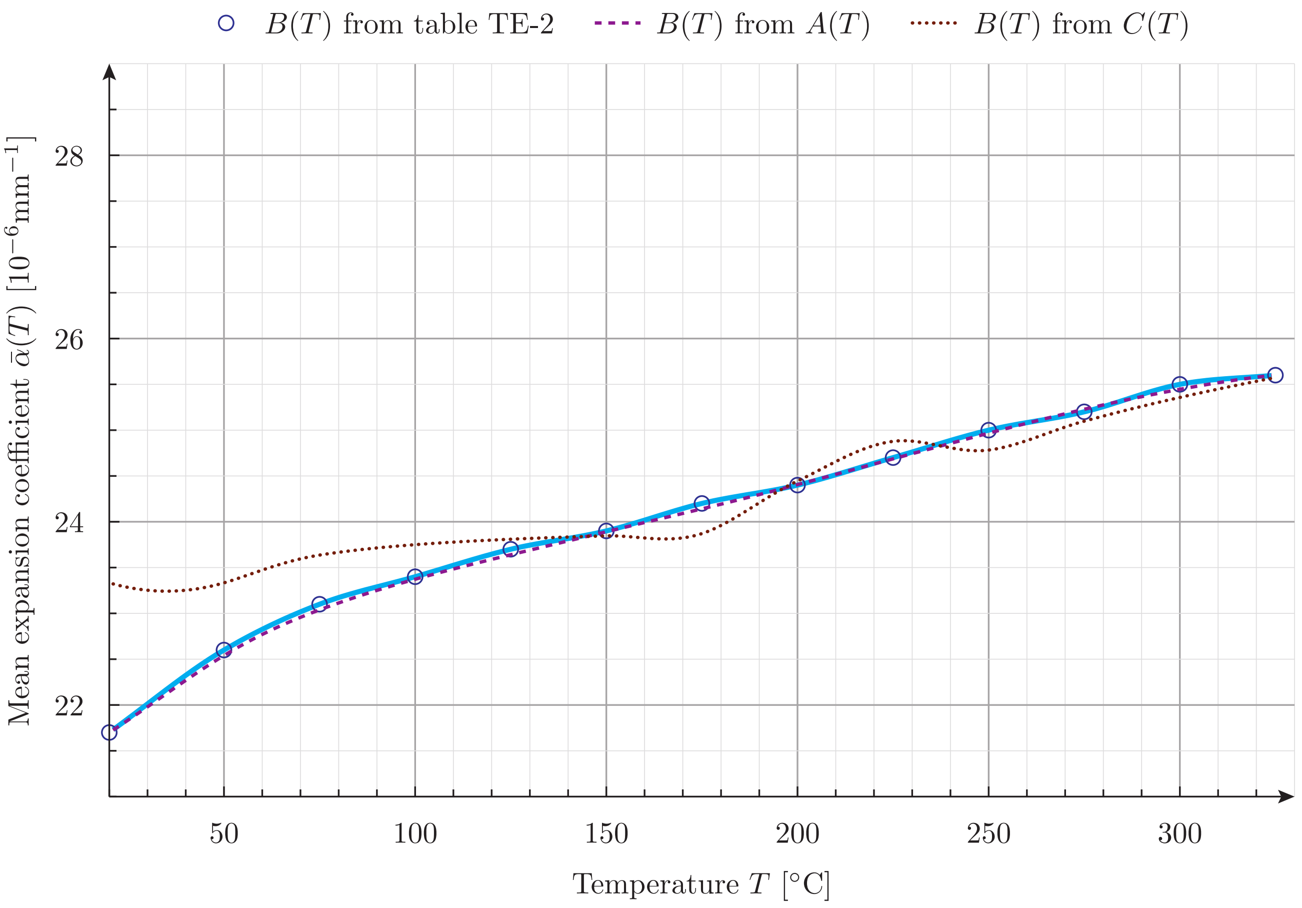
| $n$ | $L$ | $\Gamma$ | $\mu$ | $M$ |
|---|---|---|---|---|
| 1 | -6.4e-04 | -6.1e-01 | 0.2923 | 0.2923 |
| 2 | +4.0e-04 | +3.8e-01 | 0.1164 | 0.4087 |
| 3 | -3.5e-04 | -3.3e-01 | 0.0861 | 0.4948 |
| 4 | +2.3e-04 | +2.3e-01 | 0.0395 | 0.5343 |
| 5 | -2.0e-04 | -1.9e-01 | 0.0292 | 0.5634 |
| 6 | +1.4e-04 | +1.3e-01 | 0.0140 | 0.5774 |
| 7 | -1.3e-04 | -1.3e-01 | 0.0133 | 0.5908 |
| 8 | +1.1e-04 | +1.1e-01 | 0.0087 | 0.5995 |
| 9 | +1.2e-04 | +1.1e-01 | 0.0096 | 0.6091 |
| 10 | -7.3e-05 | -6.9e-02 | 0.0038 | 0.6129 |

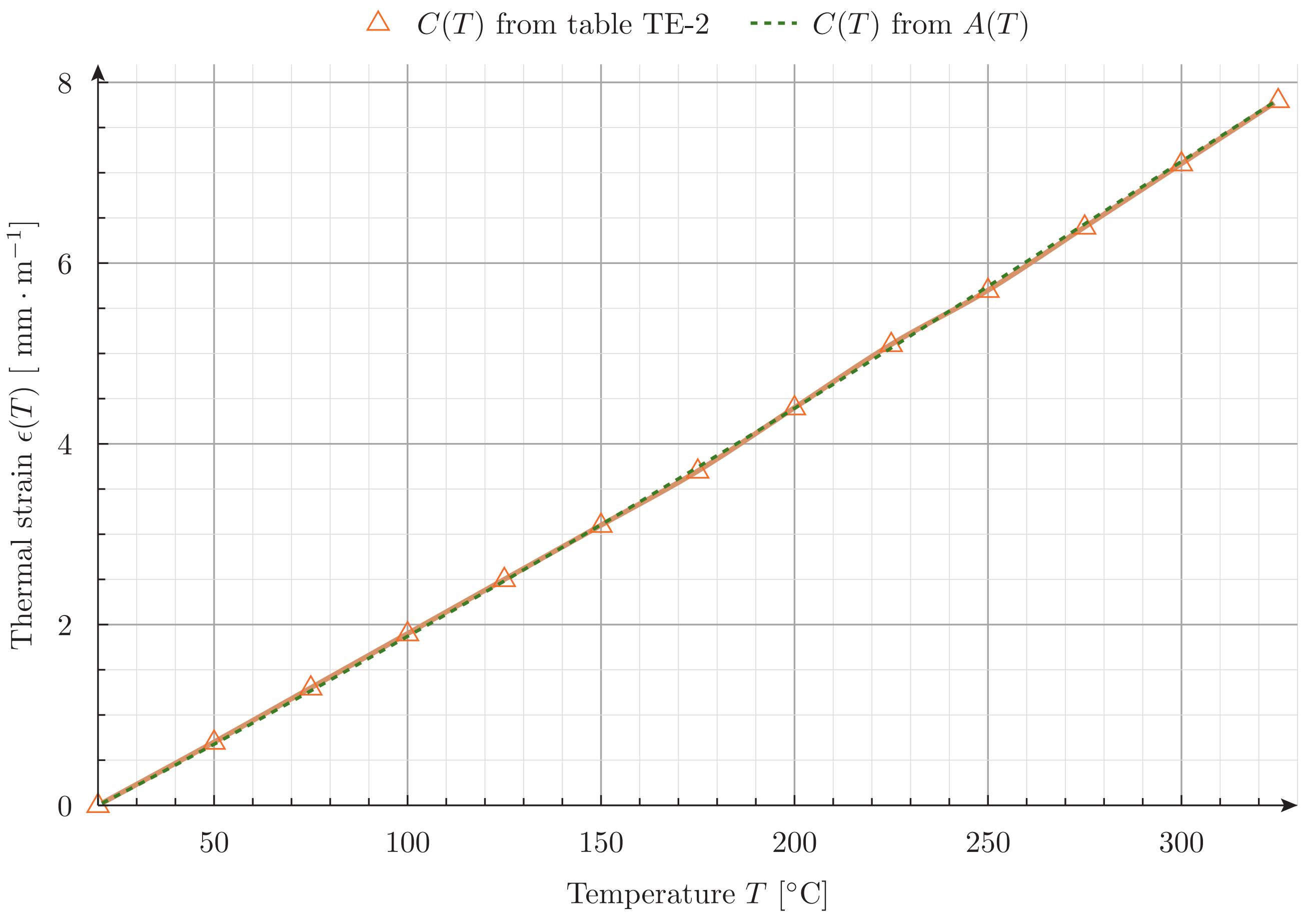
```
# just in case we wanted to interpolate with another method (linear, splines, etc.)
DEFAULT_ARGUMENT_VALUE 1 steffen
```

```
# read columns from data file and interpolate
# A is the instantaenous coefficient of thermal expansion x 10^-6 (mm/mm/ºC)
FUNCTION A(T) FILE asme-expansion-table.dat COLUMNS 1 2 INTERPOLATION $1
# B is the mean coefficient of thermal expansion x 10^-6 (mm/mm/ºC) in going
# from 20ºC to indicated temperature
FUNCTION B(T) FILE asme-expansion-table.dat COLUMNS 1 3 INTERPOLATION $1
# C is the linear thermal expansion (mm/m) in going from 20ºC
# to indicated temperature
FUNCTION C(T) FILE asme-expansion-table.dat COLUMNS 1 4 INTERPOLATION $1

VAR T'                     # dummy variable for integration
T0 = 20                    # reference temperature
T_min = vecmin(vec_A_T)    # smallest argument of function A(T)
T_max = vecmax(vec_A_T)    # largest argument of function A(T)

# compute one column from another one
A_fromC(T) := 1e3*derivative(C(T'), T', T)

B_fromA(T) := integral(A(T'), T', T0, T)/(T-T0)
B_fromC(T) := 1e3*C(T)/(T-T0)    # C is in mm/m, hence the 1e3

C_fromA(T) := 1e-3*integral(A(T'), T', T0, T)

# write interpolated results
PRINT_FUNCTION A A_fromC   B B_fromA B_fromC   C C_fromA MIN T_min+1 MAX T_max-1 STEP 1
```

```
$ cat asme-expansion-table.dat
# temp  A       B       C
20      21.7    21.7    0
50      23.3    22.6    0.7
75      23.9    23.1    1.3
100     24.3    23.4    1.9
125     24.7    23.7    2.5
150     25.2    23.9    3.1
175     25.7    24.2    3.7
200     26.4    24.4    4.4
225     27.0    24.7    5.1
250     27.5    25.0    5.7
275     27.7    25.2    6.4
300     27.6    25.5    7.1
325     27.1    25.6    7.8
$ feenox asme-expansion.fee > asme-expansion-interpolation.dat
$ pyxplot asme-expansion.ppl
$
```


□ $A(T)$ from table TE-2
$A(T)$ from $B(T)$
Instantaneous expansion coefficient $\alpha(T)$ [$10^{-6}$mm$^{-1}$]
22
24
26
28
50
100
150
200
250
300
Temperature $T$ [°C]


○ $B(T)$ from table TE-2
$B(T)$ from $A(T)$
$B(T)$ from $C(T)$
Mean expansion coefficient $\bar{\alpha}(T)$ [$10^{-6}$mm$^{-1}$]
22
24
26
28
50
100
150
200
250
300
Temperature $T$ [°C]



The conclusion (see this, this and this reports) is that values rounded to only one decimal value as presented in the ASME code section II subsection D tables are not enough to satisfy the mathematical relationships between the physical magnitudes related to thermal expansion properties of the materials listed. Therefore, care has to be taken as which of the three columns is chosen when using the data for actual thermo-mechanical numerical computations. As an exercise, the reader is encouraged to try different interpolation algorithms to see how the results change. *Spoiler alert*: they are also highly sensible to the interpolation method used to "fill in" the gaps between the table values.

```
DEFAULT_ARGUMENT_VALUE 1 steffen
DEFAULT_ARGUMENT_VALUE 2 hex

PROBLEM mechanical
READ_MESH cube-hex.msh

# aluminum-like linear isotropic material properties
E = 69e3
nu = 0.28

# free expansion
BC left   u=0
BC front  v=0
BC bottom w=0
```

```
# reference temperature is 20ºC
T0 = 20
# spatial temperature distribution symmetric wrt x,y & z
T(x,y,z) = 30+150*sqrt(x^2+y^2+z^2)

# read ASME data
FUNCTION A(T') FILE asme-expansion-table.dat COLUMNS 1 2 INTERPOLATION $1
FUNCTION B(T') FILE asme-expansion-table.dat COLUMNS 1 3 INTERPOLATION $1
FUNCTION C(T') FILE asme-expansion-table.dat COLUMNS 1 4 INTERPOLATION $1

# remember that the thermal expansion coefficients have to be
#   1. the mean value between T0 and T
#   2. functions of space, so temperature has to be written as T(x,y,z)

# in the x direction, we use column B directly
alpha_x(x,y,z) = 1e-6*B(T(x,y,z))

# in the y direction, we convert column A to mean
alpha_y(x,y,z) = 1e-6*integral(A(T'), T', T0, T(x,y,z))/(T(x,y,z)-T0)

# in the z direction, we convert column C to mean
alpha_z(x,y,z) = 1e-3*C(T(x,y,z))/(T(x,y,z)-T0)

SOLVE_PROBLEM

WRITE_MESH cube-orthotropic-expansion-$1-$2.vtk T VECTOR u v w
PRINT %.3e "displacement in x at (1,1,1) = " u(1,1,1)
PRINT %.3e "displacement in y at (1,1,1) = " v(1,1,1)
PRINT %.3e "displacement in z at (1,1,1) = " w(1,1,1)
```

```
$ gmsh -3 cube-hex.geo
[...]
$ gmsh -3 cube-tet.geo
[...]
$ feenox cube-orthotropic-expansion.fee
displacement in x at (1,1,1) =  4.451e-03
displacement in y at (1,1,1) =  4.449e-03
displacement in z at (1,1,1) =  4.437e-03
$ feenox cube-orthotropic-expansion.fee linear tet
displacement in x at (1,1,1) =  4.451e-03
displacement in y at (1,1,1) =  4.447e-03
displacement in z at (1,1,1) =  4.438e-03
$ feenox cube-orthotropic-expansion.fee akima hex
displacement in x at (1,1,1) =  4.451e-03
displacement in y at (1,1,1) =  4.451e-03
displacement in z at (1,1,1) =  4.437e-03
$ feenox cube-orthotropic-expansion.fee splines tet
displacement in x at (1,1,1) =  4.451e-03
displacement in y at (1,1,1) =  4.450e-03
displacement in z at (1,1,1) =  4.438e-03
$
```

Differences cannot be seen graphically, but they are there as the terminal mimic illustrates. Yet, they are not as large nor as sensible to meshing and interpolation settings as one would have expected after seeing the plots

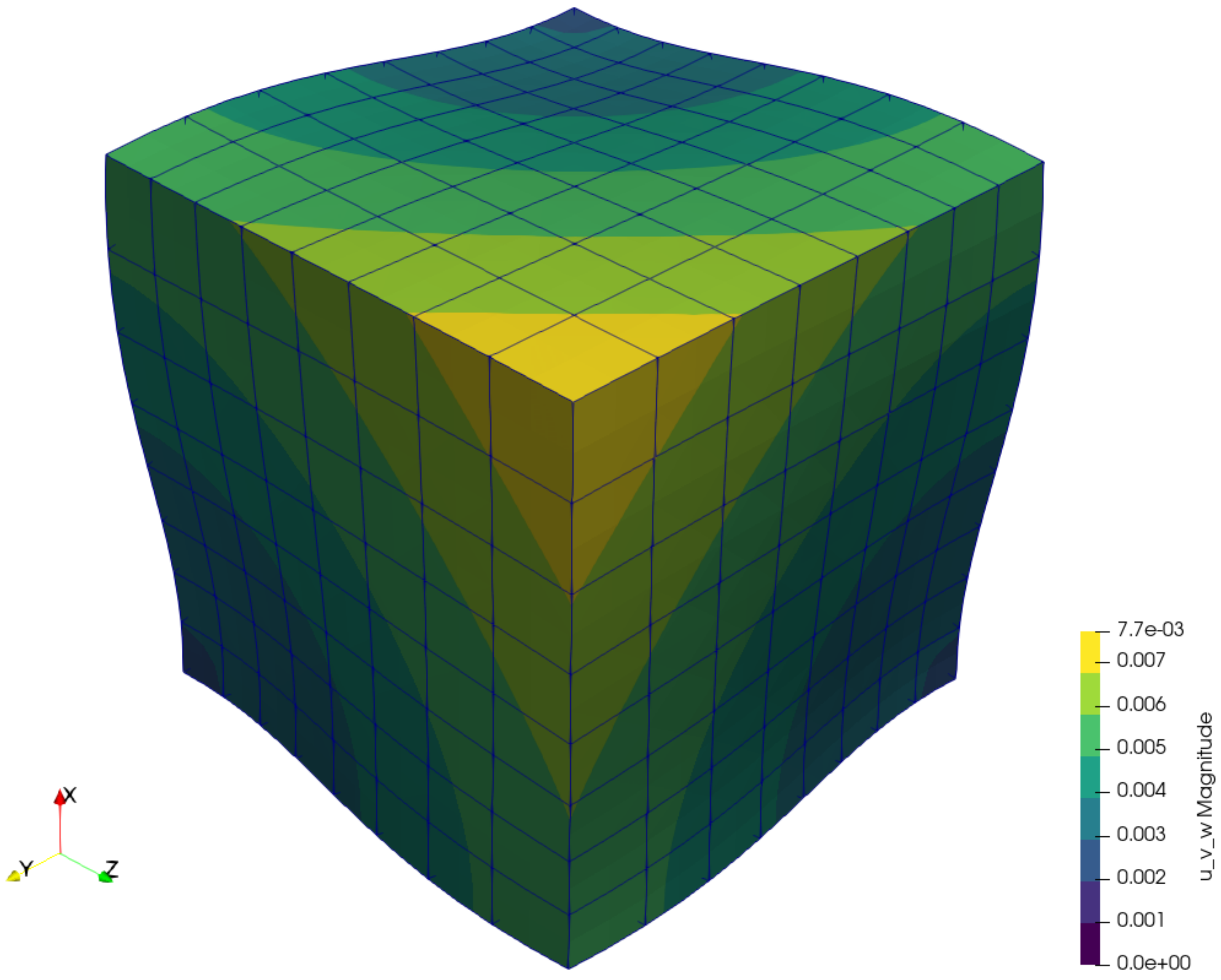


Figure 18: Warped displacement ($\times 500$) of the cube using ASME's three columns.

from the previous section.

```
PROBLEM mechanical
READ_MESH veeder.msh

b = 1     # cylinder radius
h = 0.5   # cylinder height

E = 1         # young modulus (does not matter for the displacement, only for stresses)
nu = 1/3      # poisson ratio
alpha = 1e-5  # temperature expansion coefficient

# temperature distribution as in the original paper
T1 = 1        # maximum temperature
T0 = 0        # reference temperature (where expansion is zero)
T(x,y,z) := T0 + T1*(1-(x^2+y^2)/(b^2))


# boundary conditions (note that the cylinder can still expand on the x-y plane)
BC inf      tangential radial

# solve!
SOLVE_PROBLEM

# write vtk output
WRITE_MESH veeder.vtk    T sigma dudx dudy dudz dvdx dvdy dvdz dwdx dwdy dwdz  sigma1 sigma2 sigma3   ↩
     VECTOR u v w

# non-dimensional numerical displacement profiles
v_num(z') = v(0, b, z'*h)/(alpha*T1*b)
w_num(r') = w(0, r'*b, h)/(alpha*T1*b)

########
# reference solution
# coefficients of displacement functions for h/b = 0.5
a00 =  0.66056
a01 = -0.44037
a10 =  0.23356
a02 = -0.06945
a11 = -0.10417
a20 =  0.00288

b00 = -0.01773
b01 = -0.46713
b10 = -0.04618
b02 = +0.10417
b11 = -0.01152
b20 = -0.00086

# coefficients of displacement functions for h/b = 1.0
# a00 =  0.73197
# a01 = -0.48798
# a10 =  0.45680
# a02 = -0.01140
# a11 = -0.06841
# a20 =  0.13611
#
# b00 =  0.26941
# b01 = -0.45680
# b10 = -0.25670
# b02 =  0.03420
# b11 = -0.27222
# b20 = -0.08167


R(r') = r'^2 - 1
Z(z') = z'^2 - 1

v_ref(r',z') = r' * (a00 + a01*R(r') + a10*Z(z') + a02* R(r')^2 + a11 * R(r')*Z(z') + a20 * Z(z')^2)
w_ref(r',z') = z' * (b00 + b01*R(r') + b10*Z(z') + b02* R(r')^2 + b11 * R(r')*Z(z') + b20 * Z(z')^2)


PRINT_FUNCTION FILE veeder_v.dat  v_num v_ref(1,z') MIN 0 MAX 1 NSTEPS 50 HEADER
PRINT_FUNCTION FILE veeder_w.dat  w_num w_ref(r',1) MIN 0 MAX 1 NSTEPS 50 HEADER
```

```
$ gmsh -3 veeder.geo
[...]
$ feenox veeder.fee
$ pyxplot veeder.ppl
$
```

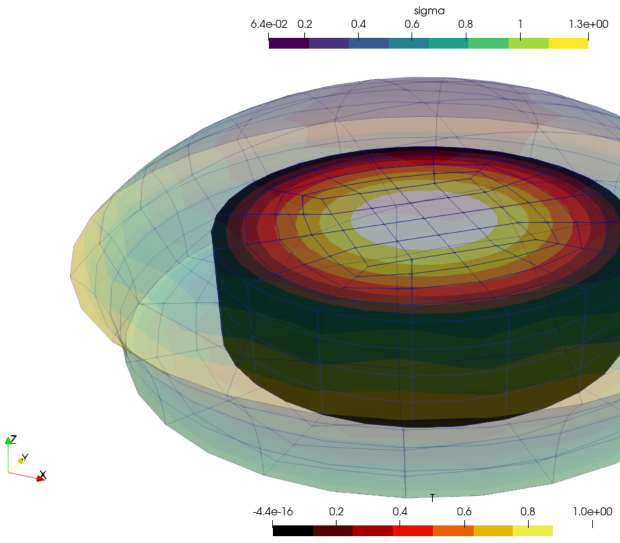


Figure 19: 100,000x-warped displacements

```
# 2d plane strain mechanical problem over the [-1:+1]x[-1:+1] square
PROBLEM mechanical plane_strain
READ_MESH square-centered.msh

# fixed at left, uniform traction in the x direction at right
BC left    fixed
BC right   tx=50

# ASME II Part D pag. 785 Carbon steels with C<=0.30%
FUNCTION E_carbon(temp) INTERPOLATION steffen DATA {
-200  216
-125  212
```

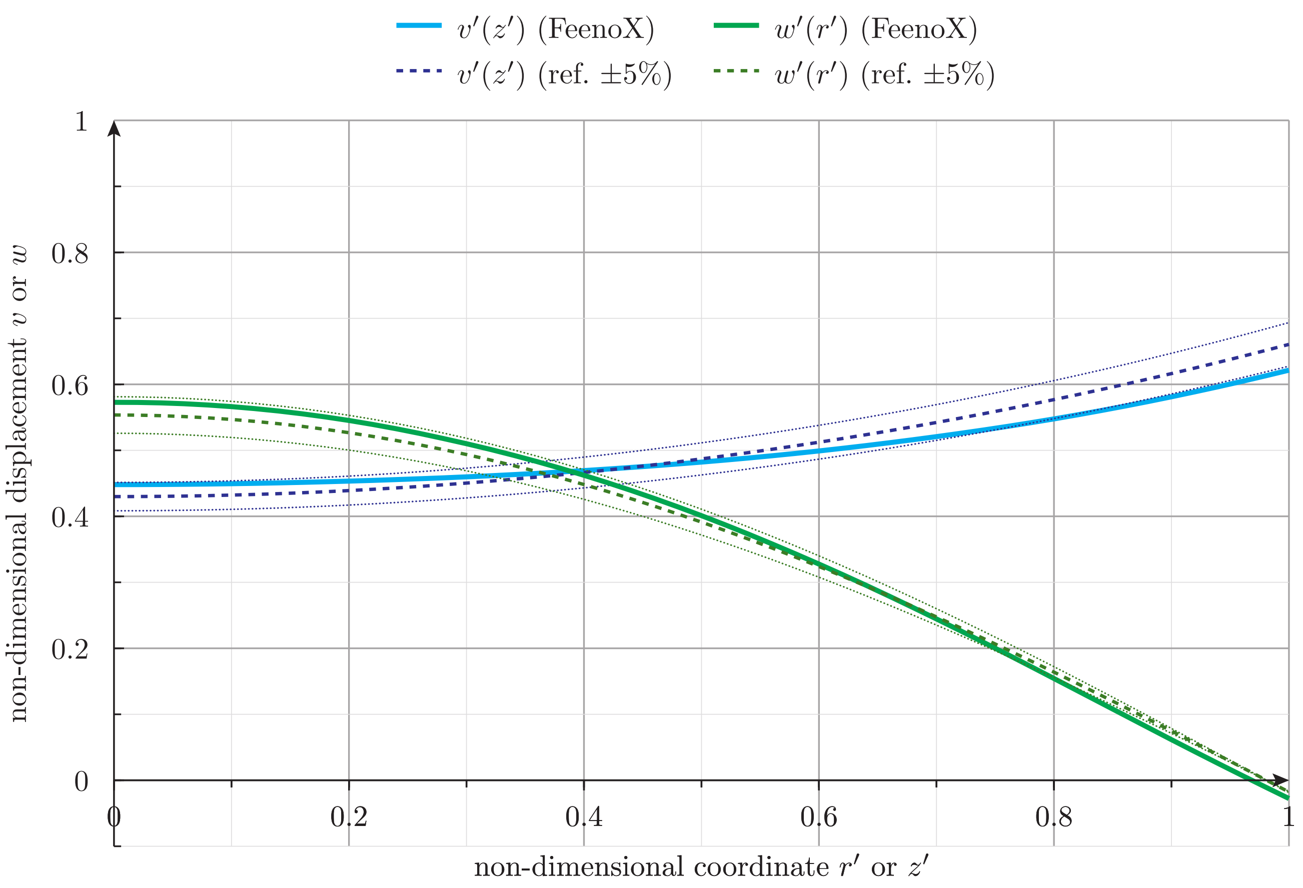


Figure 20: Comparison of 1-D displacement profiles

```
-75   209
25    202
100   198
150   195
200   192
250   189
300   185
350   179
400   171
450   162
500   151
550   137
}

# read the temperature according to the run-time argument $1
INCLUDE mechanical-square-temperature-$1.fee

# Young modulus is the function above evaluated at the local temperature
E(x,y) := E_carbon(T(x,y))

# uniform Poisson's ratio
nu = 0.3

SOLVE_PROBLEM
PRINT u(1,1) v(1,1)
WRITE_MESH mechanical-square-temperature-$1.vtk  E T VECTOR u v 0
```

```
$ gmsh -2 square-centered.geo
[...]
Info    : Done meshing 2D (Wall 0.00117144s, CPU 0.00373s)
Info    : 1089 nodes 1156 elements
Info    : Writing 'square-centered.msh'...
Info    : Done writing 'square-centered.msh'
Info    : Stopped on Thu Aug  4 09:40:09 2022 (From start: Wall 0.00818854s, CPU 0.031239s)
$ feenox mechanical-square-temperature.fee uniform
0.465632        -0.105128
$ feenox mechanical-square-temperature.fee linear
0.589859        -0.216061
$ gmsh -2 square-centered-unstruct.geo
[...]
Info    : Done meshing 2D (Wall 0.0274833s, CPU 0.061072s)
Info    : 65 nodes 132 elements
Info    : Writing 'square-centered-unstruct.msh'...
Info    : Done writing 'square-centered-unstruct.msh'
Info    : Stopped on Sun Aug  7 18:33:41 2022 (From start: Wall 0.0401667s, CPU 0.107659s)
$ feenox thermal-square.fee
$ feenox mechanical-square-temperature.fee mesh
0.589859        -0.216061
$
```

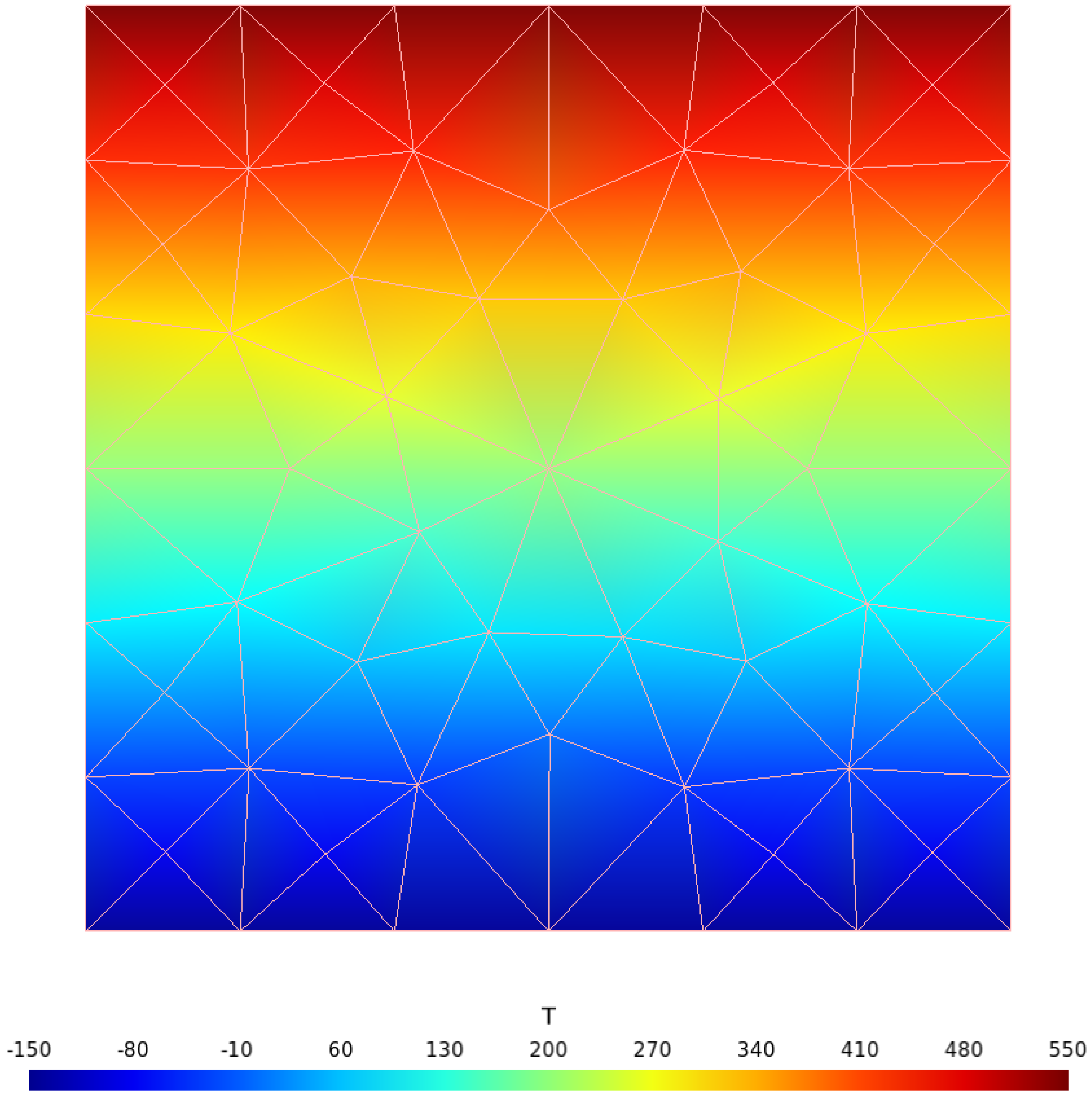


Figure 21: Temperature distribution from a heat conduction problem.

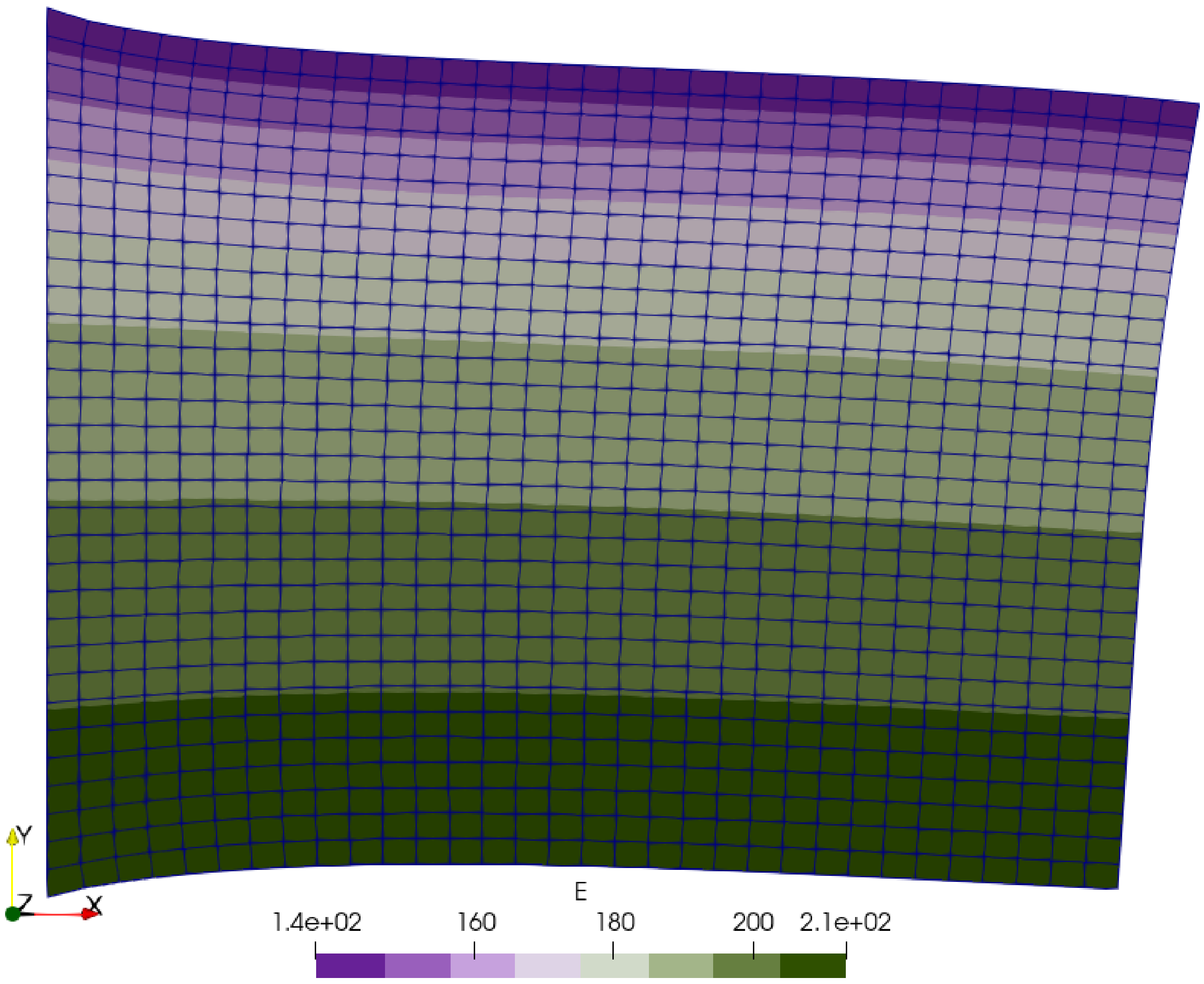


Figure 22: Young modulus distribution over the final displacements.