Solving mazes with PDEs instead of AI

FeenoX Tutorial #2

Contents

1 Foreword

Welcome the the **Second Feenox tutorial**, very much inspired on Donald Knuth's Selected Papers on Fun and Games. And a little bit on Homer Simpson.

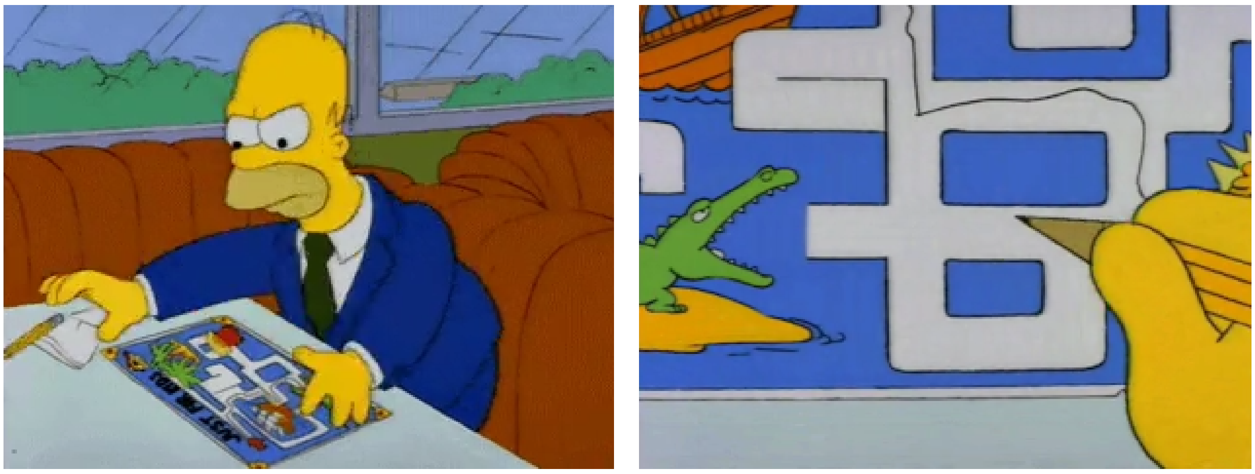

Figure 1: Homer trying to solve a maze on a placemat

Say you want to solve a maze drawn in a restaurant's placemat: one where both the start and end points are known beforehand as shown in fig. 1. Nowadays the first instinctive reaction would be to fire some fancy AI or ML algorithm. But we are actual engineers and we do value nice maths hacks, don't we? In order to avoid falling into the alligator's mouth, we can exploit the ellipticity of the Laplacian operator to solve any maze—even hand-drawn ones. Just FeenoX and a bunch of standard open source tools to convert a bitmapped picture of the maze into an unstructured mesh.

> A couple of LinkedIn posts to see some comments and discussions:
>
> - https://www.linkedin.com/feed/update/urn:li:activity:6831291311832760320/
> - https://www.linkedin.com/feed/update/urn:li:activity:6973982270852325376/
>
> Other people's maze-related posts:
>
> - https://www.linkedin.com/feed/update/urn:li:activity:7117082910381232128/
> - https://www.linkedin.com/feed/update/urn:li:activity:6972370982489509888/
> - https://www.linkedin.com/feed/update/urn:li:activity:6972949021711630336/
> - https://www.linkedin.com/feed/update/urn:li:activity:6973522069703516160/
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> - https://www.linkedin.com/feed/update/urn:li:activity:6974663157952745472/
> - https://www.linkedin.com/feed/update/urn:li:activity:6974979951049519104/
> - https://www.linkedin.com/feed/update/urn:li:activity:6982049404568449024/

1.1 Executive summary

We will…

1. Create and download a random maze online.
2. Create a FEM mesh for the aisle of the maze (the walls will be the boundary of the domain) using Gmsh.

3. Solve $\nabla^2\phi(x,y)=0$ subject to the following boundary conditions:

$$\begin{cases} \phi(x,y)=1 & \text{at the inlet} \\ \phi(x,y)=0 & \text{at the outlet} \\ \frac{\partial\phi}{\partial n}=0 & \text{everywhere else} \end{cases}$$

```
PROBLEM laplace 2D
READ_MESH maze.msh

BC start  phi=0
BC end    phi=1

SOLVE_PROBLEM
```

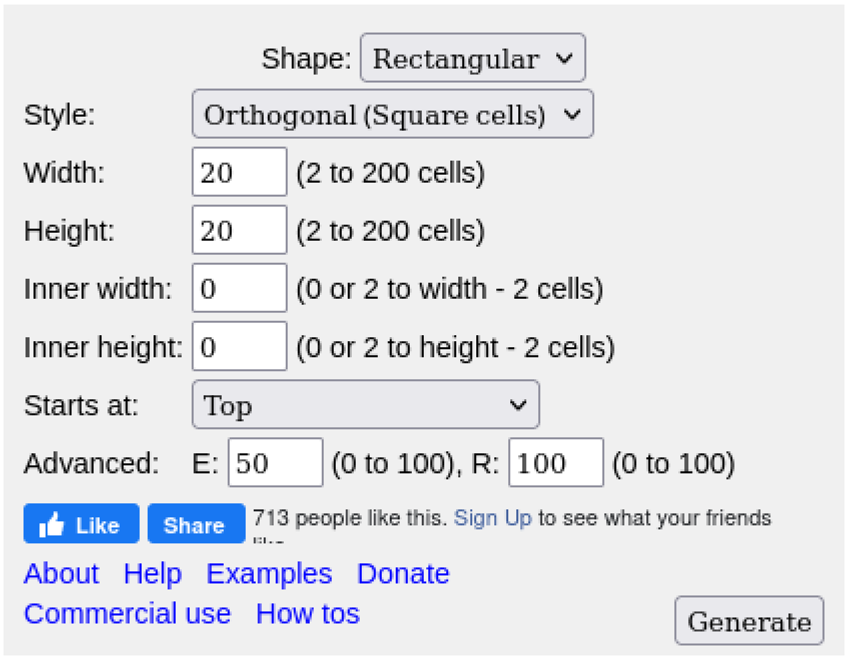
4. Show the solution of the maze given by $\nabla\phi$, properly scaled, in Gmsh:

```
# write the norm of gradient as a scalar field
# and the gradient as a 2d vector into a .msh file
 WRITE_MESH maze-solved.msh \
     sqrt(dphidx(x,y)^2+dphidy(x,y)^2) \
     VECTOR dphidx dphidy 0
```

5. As a bonus, we will also solve a transient case to see how Laplace "tries" all the paths and only keeps the ones that do not find any dead end.

2 Creating the mesh

1. Go to http://www.mazegenerator.net/
2. Choose shape, style, width, height, etc. and create a maze


Shape:	Rectangular
Style:	Orthogonal (Square cells)
Width:	20 (2 to 200 cells)
Height:	20 (2 to 200 cells)
Inner width:	0 (0 or 2 to width - 2 cells)
Inner height:	0 (0 or 2 to height - 2 cells)
Starts at:	Top
Advanced:	E: 50 (0 to 100), R: 100 (0 to 100)

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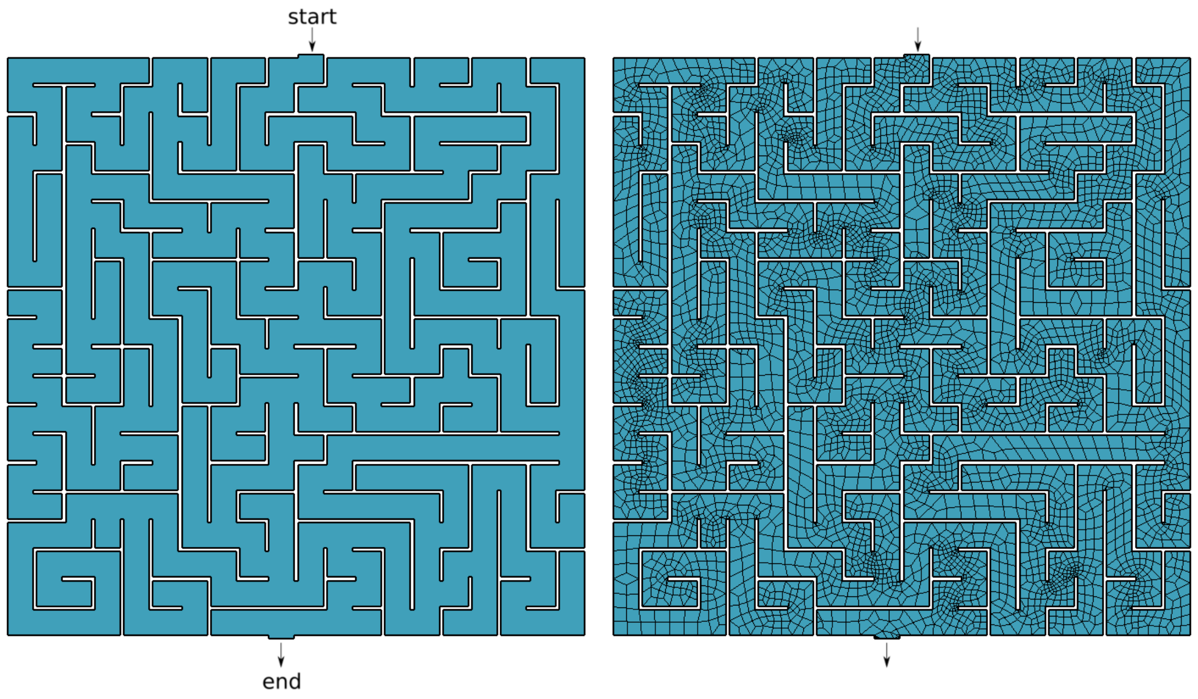
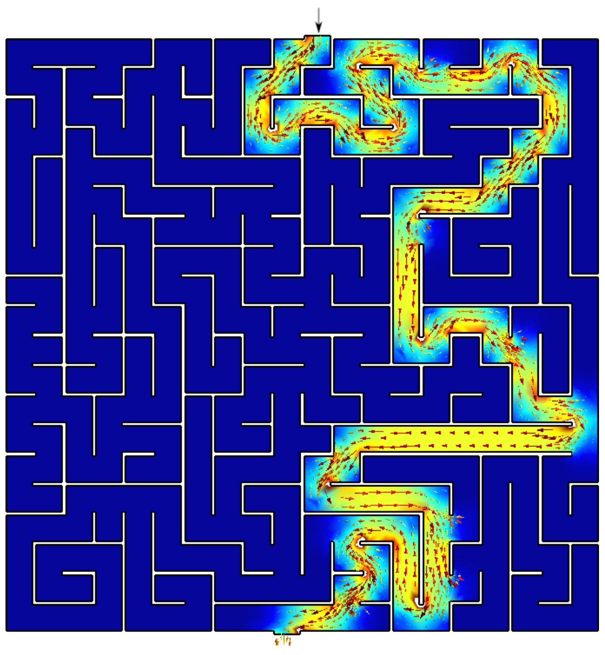


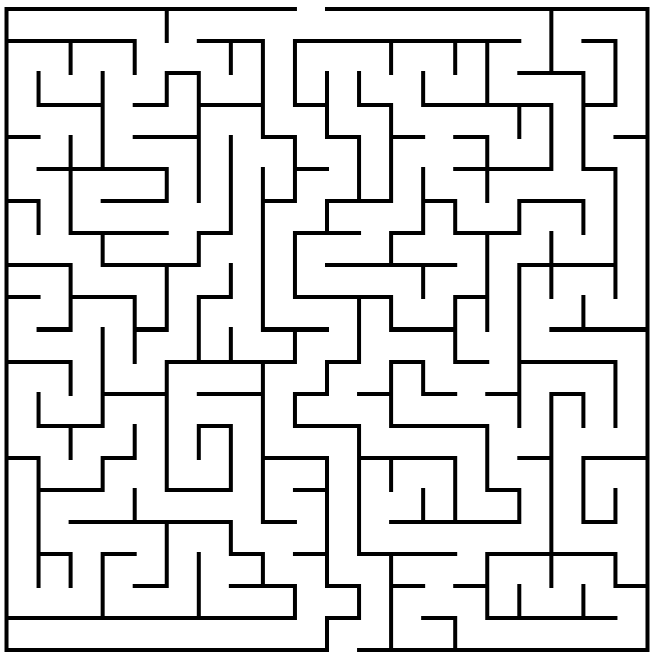
(a) Bitmapped maze from https://www.mazegenerator.net (left) and 2D mesh (right)



(b) Solution to found by FeenoX (and drawn by Gmsh)

Figure 2: Bitmapped, meshed and solved mazes.

3. Download it in PNG (fig. 2a)

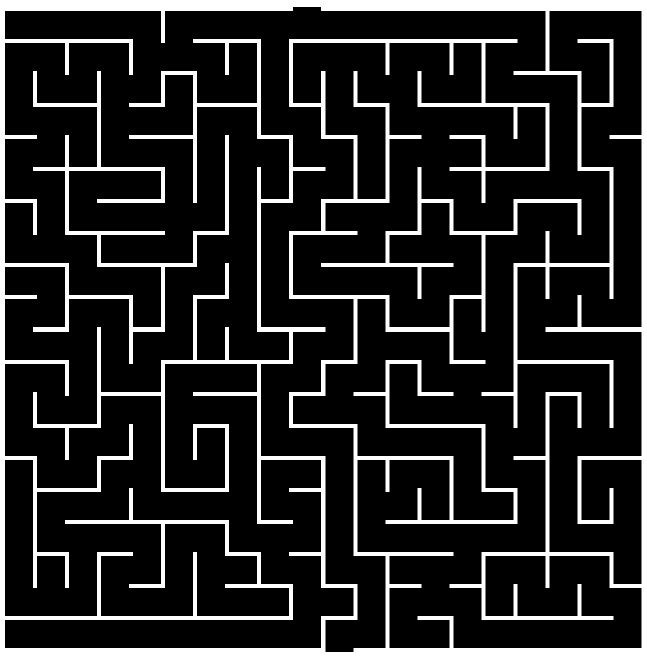


4. Perform some manipulations and conversions. We are going to need a bunch of free an open source tools:

```
sudo apt-get install imagemagick potrace inkscape pstoedit make g++
```

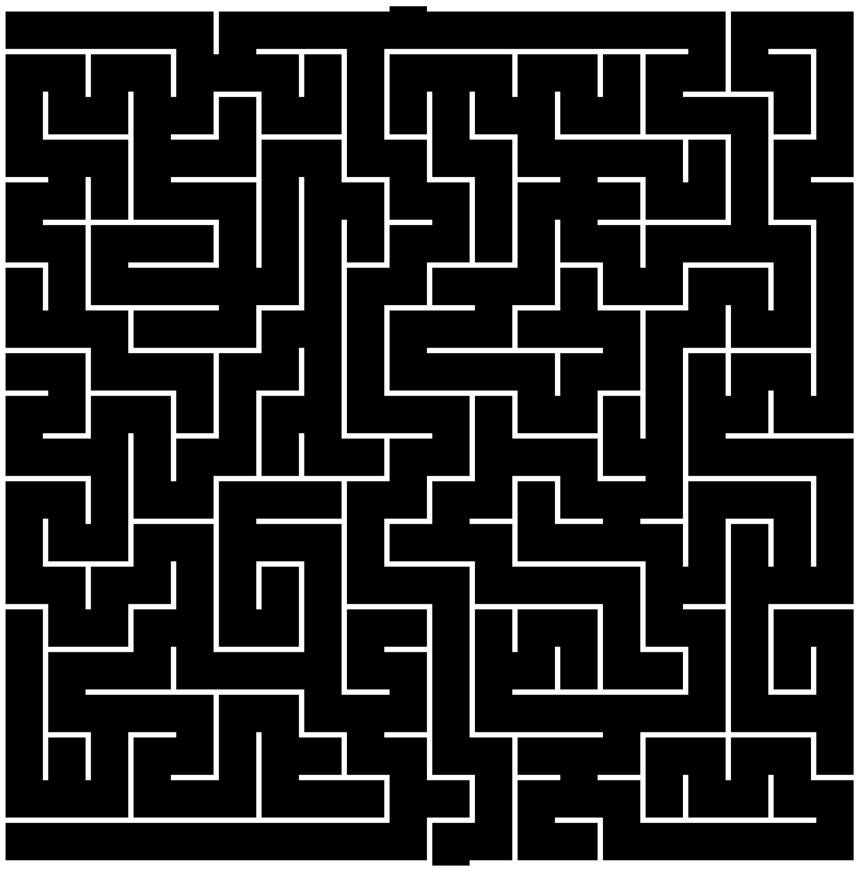
   a. Invert the PNG and convert it to PNM with `convert` so the aisles are black and the walls are white:

   ```
   convert maze.png -negate maze_negated.pnm
   ```



b. Vectorize the negated bitmap into an SVG with `potrace`:

```
potrace maze_negated.pnm --alphamax 0  --opttolerance 0 -b svg -o maze.svg
```



c. Convert the SVG to EPS with `inskcape`

```
inkscape maze.svg --export-eps=maze.eps
```

d. Convert the EPS to DXF with `pstoedit`

```
pstoedit -dt -f 'dxf:-polyaslines -mm' maze.eps > maze.dxf
```

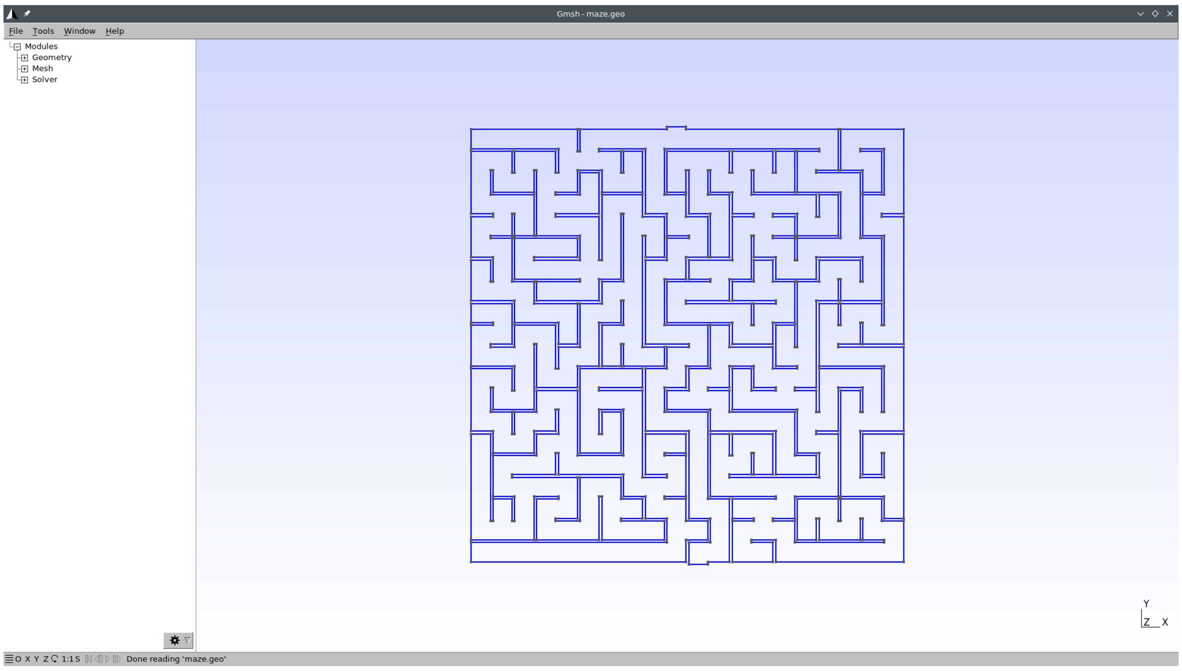
e. Convert the DXF to a Gmsh's `.geo` with `dxf2geo` (a C++ tool that comes with Gmsh)

```
make dxf2geo
./dxf2geo maze.dxf 0.1
```

The tutorial directory contains a convenience script to perfom all these steps in a single call. Just name your PNG `maze.png` and run `./png2geo.sh`.

5. The last file `maze.geo` contains only lines with the walls of the maze. We need to tell FeenoX where the maze starts and where it ends. Open it with Gmsh:

```
gmsh maze.geo
```

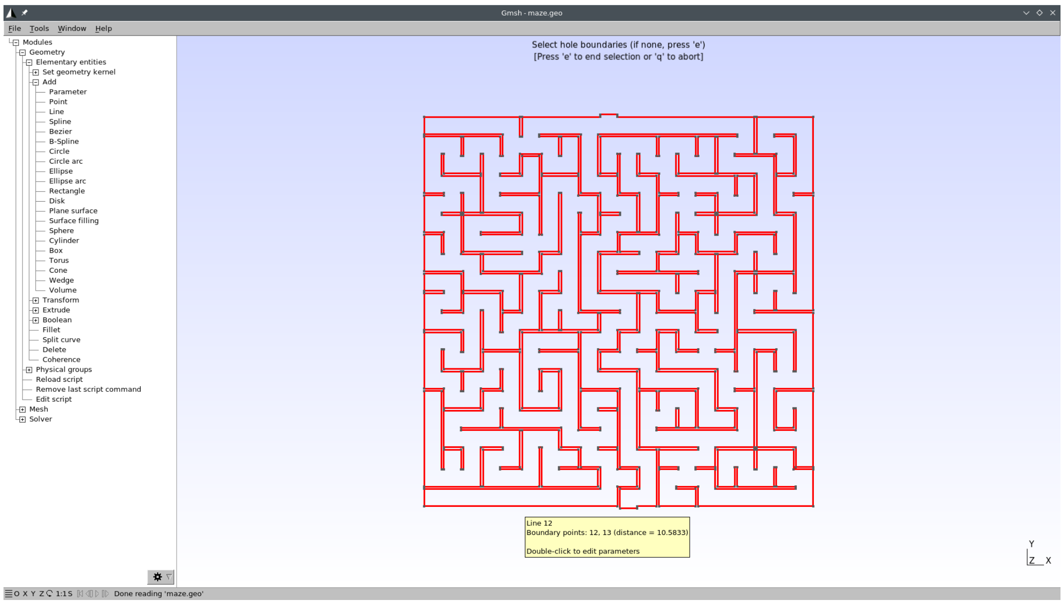


Done reading 'maze.geo'

We have to

i. Add a surface for the aisles of the maze (i.e. the 2D domain where we will apply the Laplace operator):

- In the left-pane tree, go to Geometry → Elementary entities → Add → Plane Surface.
- When prompted to "select surface boundary" click on any of the blue edges. They all should turn to red:



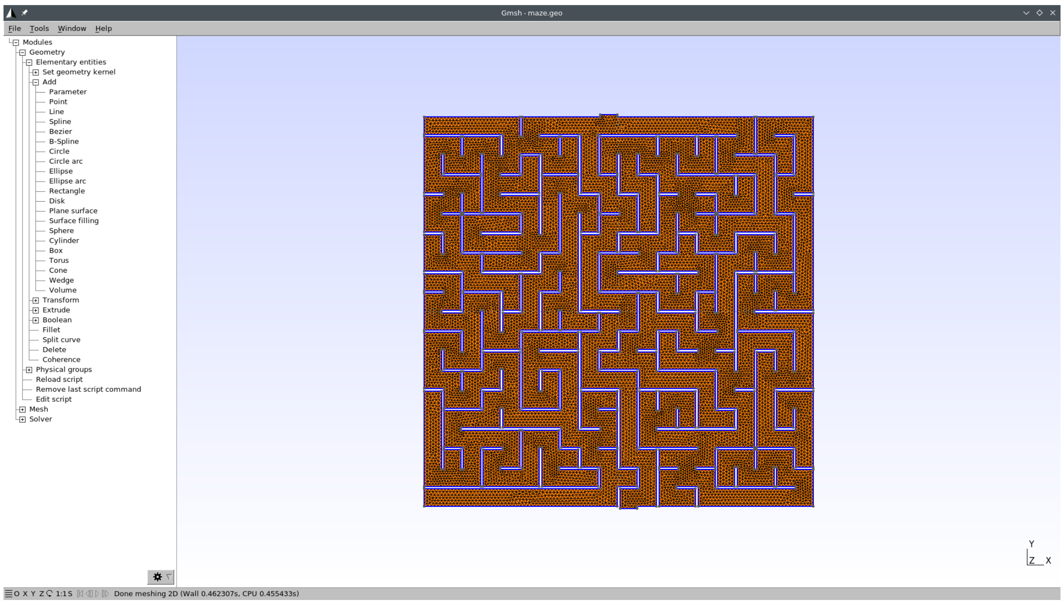
Select hole boundaries (if none, press 'e')
[Press 'e' to end selection or 'q' to abort]

Line 12
Boundary points: 12, 13 (distance = 10.5833)

Double-click to edit parameters

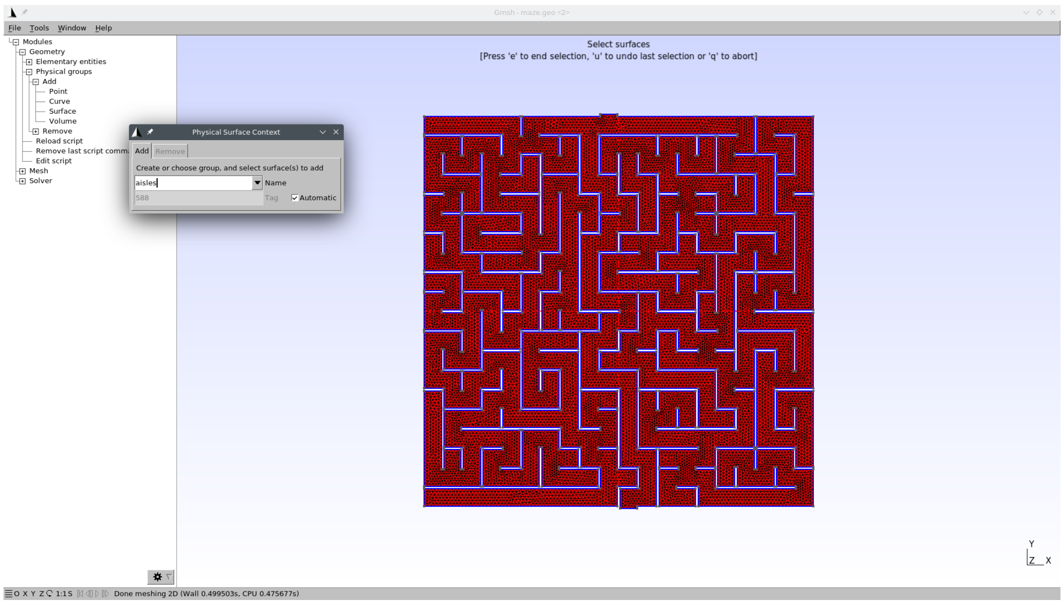
Done reading 'maze.geo'

- Press `e` to end selection and `q` to exit the selection mode.
- Press `2` to make a preliminary 2D mesh to see if it worked:



```
Done meshing 2D (Wall 0.462307s, CPU 0.455433s)
```

- Go to Geometry → Physical groups → Add → Surface.
- When prompted to "select surfaces" click on any point of the maze. The mesh should turn red:



```
Select surfaces
[Press 'e' to end selection, 'u' to undo last selection or 'q' to abort]
```

Create or choose group, and select surface(s) to add

Name: aisles

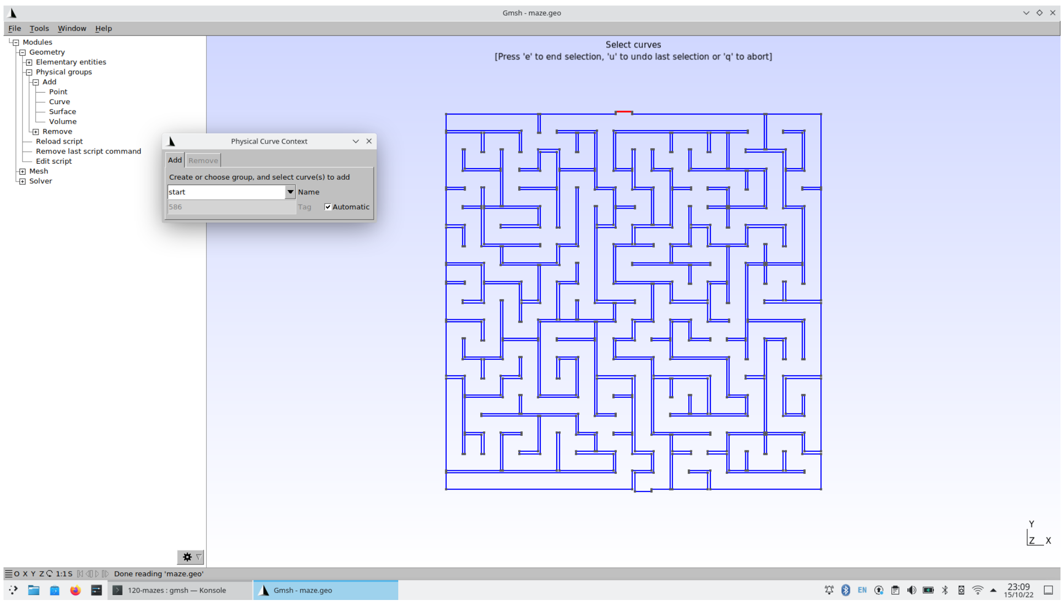
Tag: 588 ☑ Automatic

```
Done meshing 2D (Wall 0.499503s, CPU 0.475677s)
```

- Write "aisles" as the group's name.
- Press `e` to end the selection. You might need to click somewhere in the Gmsh window so as to get the focus out of the name text field.
- Press `q` to exit the selection mode.
- Press `0` to reload the `maze.geo` file and remove the temporary mesh.

ii. Set physical curves for "start" and "end" (i.e. the 1D edges that will hold the Dirichlet boundary conditions)

- In the left-pane tree, go to Geometry → Physical groups → Add → Curve.
- When prompted to "select curves" click on the edge (or edges) that define the inlet.
- Write "start" as the group's name.
- Press `e` to end the selection. You might need to click somewhere in the Gmsh window so as to get the focus out of the name text field.


Gmsh - maze.geo

Select curves
[Press 'e' to end selection, 'u' to undo last selection or 'q' to abort]

Physical Curve Context

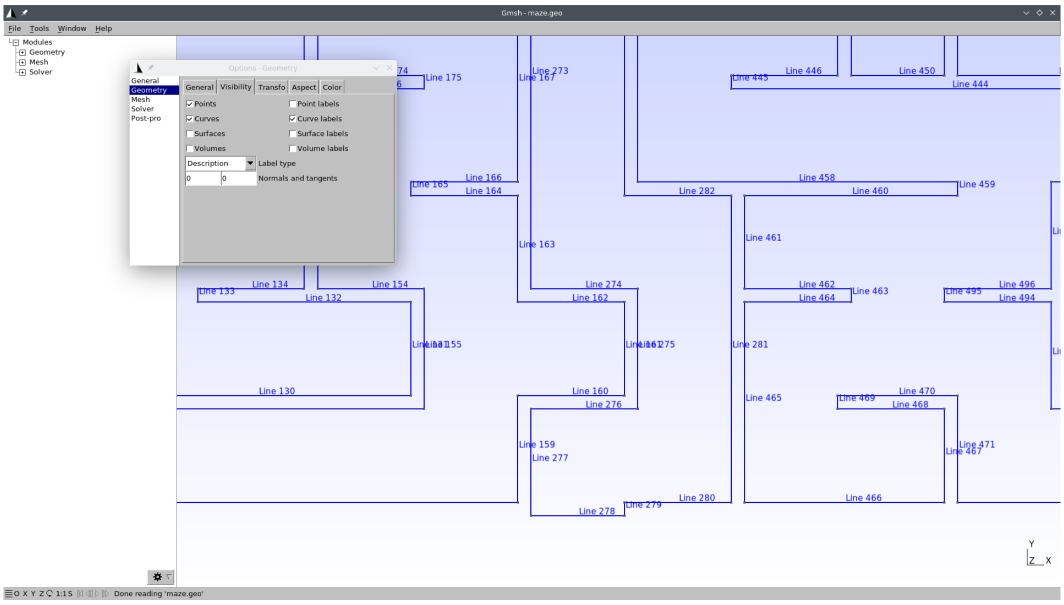
Create or choose group, and select curve(s) to add

start — Name

586 — Tag — ✓ Automatic

Done reading 'maze.geo'

- Do the same thing for the outlet, and name it "end." Alternatively, find out the edge id of the outlet by zooming in a little bit into the outlet and then going to Tools → Options → Geometry → Curve Labels:


Gmsh - maze.geo

Options - Geometry

✓ Points — Point labels
✓ Curves — ✓ Curve labels
Surfaces — Surface labels
Volumes — Volume labels
Description — Label type
0 — 0 — Normals and tangents

Done reading 'maze.geo'

In this case, the curve id is 278. So we can edit the file `maze.geo` and add this line at the very end:

```
Physical Curve("end") = {278};
```

Now our `maze.geo` contains a proper definition of the domain where we will solve $\nabla^2\phi = 0$.

6. Create the actual mesh `maze.msh` out of `maze.geo`

```
gmsh -2 maze.geo
```

3 Solving the steady-state Laplace equation

We have to solve $\nabla^2\phi = 0$ with the following boundary conditions

$$\begin{cases} \phi = 0 & \text{at "start"} \\ \phi = 1 & \text{at "end"} \\ \nabla\phi \cdot \hat{\mathbf{n}} = 0 & \text{everywhere else} \end{cases}$$

The solution to the maze is given by the gradient of the solution $\nabla\phi$.

This translates into FeenoX to the following input file:

```
PROBLEM laplace 2D  # pretty self-descriptive, isn't it?
READ_MESH maze.msh

# boundary conditions (default is homogeneous Neumann)
BC start  phi=0
BC end    phi=1

SOLVE_PROBLEM

# write the norm of gradient as a scalar field
# and the gradient as a 2d vector into a .msh file
WRITE_MESH maze-solved.msh \
    sqrt(dphidx(x,y)^2+dphidy(x,y)^2) \
    VECTOR dphidx dphidy 0
```

Let us break the `WRITE_MESH` instruction up. The resulting mesh name is `maze-solved.msh`. It cannot be the same `maze.msh` we used as the input mesh.

Those aisles that go into dead ends are expected to have a very small gradient, while the aisles that lead to the exit are expected to have a large gradient. The solution $\phi(x,y)$ is mapped into a function `phi(x,y)`. Its gradient

$$\nabla\phi = \begin{bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{bmatrix}$$

is mapped into two scalar functions, `dphidx(x,y)` and `dphidy(x,y)`. So we write a scalar field with the magnitude of the gradient $\nabla\phi$ as `sqrt(dphidx(x,y)^2+dphidy(x,y)^2)`(remember that everything is an expression, including the fields written in the post-processing files). We also write the gradient itself as a vector so we can follow the arrows from the start down to the end.

```
$ feenox maze.fee
$
```

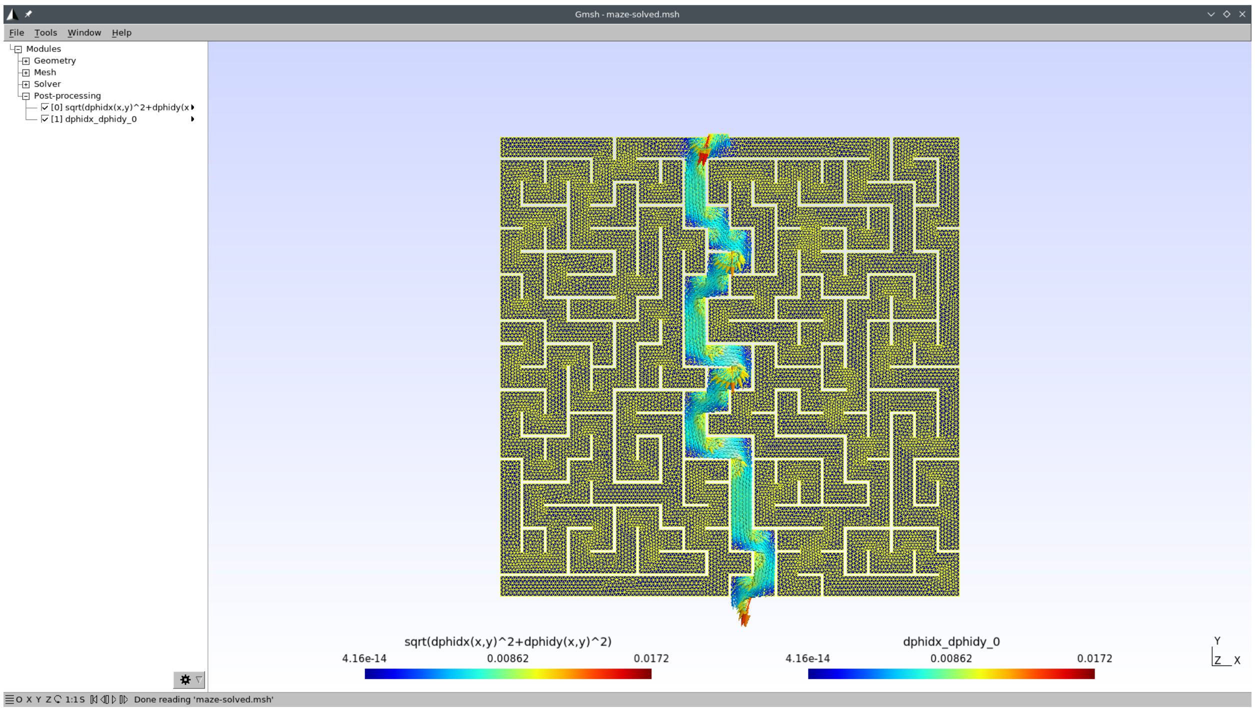
That's it. Remember the Unix rule of silence.

4 Results

Open `maze-solved.msh` with Gmsh:

```
gmsh maze-solved.msh
```

You should see something like this:



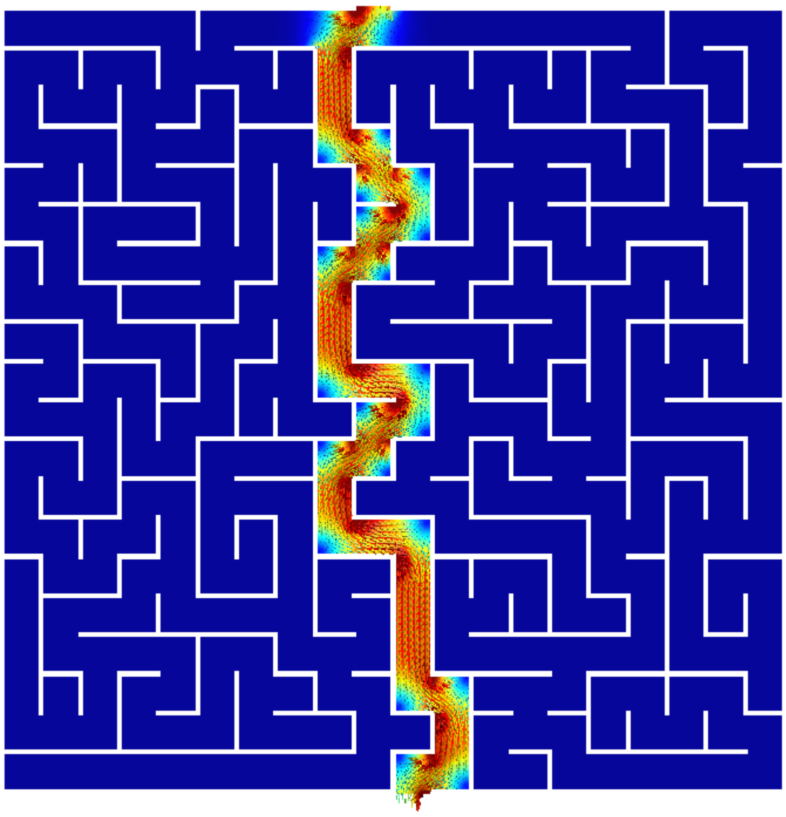
Done reading 'maze-solved.msh'

I bet you did not see the straightforward path that FeenoX found! Now we just need some make up to make a nice social-network-worth picture. Let us create a file `maze-fig.geo` so Gmsh can scale everything up for us:

```
Merge "maze-solved.msh";

Mesh.SurfaceFaces = 0;
Mesh.SurfaceEdges = 0;
General.SmallAxes = 0;

General.GraphicsHeight = 940;
General.GraphicsWidth = 1380;

View[0].ShowScale = 0;
View[0].RangeType = 2;
View[0].CustomMax = 0.008;
View[0].SaturateValues = 1;


View[1].ShowScale = 0;
View[1].CustomMax = 1e-2;
View[1].RangeType = 2;
View[1].GlyphLocation = 1; // Glyph (arrow, number, etc.) location (1: center of gravity, 2: node)
```

```
View[1].ArrowSizeMax = 20;

Print "maze-solved.png";
Exit;
```

```
$ gmsh maze-fig.geo
$
```



5 Transient Laplace

Use this transient input file for FeenoX, with the same mesh, to solve for the transient Laplace problem going from top to down (i.e. `td`):

$$\nabla^2 \phi = \alpha \frac{\partial \phi}{\partial t}$$

```
PROBLEM laplace 2D
READ_MESH maze.msh

phi_0(x,y) = 0              # inital condition
end_time = 500              # some end time where we know we reached the steady-state
alpha = 1e-3                # factor of the time derivative to make it advance faster
BC start   phi=if(t<1,t,1)  # a ramp from zero to avoid discontinuities with the initial condition
BC end     phi=0            # homogeneous BC at the end (so we move from top to bottom)

SOLVE_PROBLEM
PRINT t

WRITE_MESH maze-tran-td.msh phi sqrt(dphidx(x,y)^2+dphidy(x,y)^2) VECTOR -dphidx(x,y) -dphidy(x,y) 0
```

```
$ feenox maze-tran-td.fee
0
0.00736111
0.017873
0.0338879
0.0541553
[...]
390.125
419.195
450.667
475.333
500
$
```

So we have transient data in `maze-tran-td.msh`. Let's create a video (or a GIF) out of it. Use this `maze-tran-td-anim.geo` to create PNGs of each time step:

```
Merge "maze-tran-td.msh";

Mesh.SurfaceFaces = 0;
Mesh.SurfaceEdges = 0;
General.SmallAxes = 0;

General.GraphicsHeight = 1024;
General.GraphicsWidth = General.MenuWidth + 1024;

View[0].Visible = 0;

View[1].ShowScale = 0;
View[1].RangeType = 2;
View[1].CustomMax = 0.01;
View[1].SaturateValues = 1;


View[2].ShowScale = 0;
View[2].CustomMax = 0.005;
View[2].RangeType = 2;
View[2].GlyphLocation = 1; // Glyph (arrow, number, etc.) location (1: center of gravity, 2: node)
View[2].ArrowSizeMax = 20;

For step In {0:View[0].NbTimeStep-1}
  View[1].TimeStep = step;
  View[2].TimeStep = step;
  Print Sprintf("maze-tran-td-%03g.png", step);
  Draw;
EndFor
```

```
General.Terminal = 1;
Printf("# all frames dumped, now run");
Printf("ffmpeg -y -framerate 10 -f image2 -i maze-tran-td-%%03d.png maze-tran-td.mp4");
Printf("ffmpeg -y -framerate 10 -f image2 -i maze-tran-td-%%03d.png maze-tran-td.gif");
Exit;
```

```
$ gmsh maze-tran-td-anim.geo
# all frames dumped, now run
ffmpeg -y -framerate 10 -f image2 -i maze-tran-td-%03d.png maze-tran-td.mp4
ffmpeg -y -framerate 10 -f image2 -i maze-tran-td-%03d.png maze-tran-td.gif
$ sudo apt-get install ffmpeg
[...]
$ ffmpeg -y -framerate 10 -f image2 -i maze-tran-td-%03d.png maze-tran-td.mp4
[...]
$ ffmpeg -y -framerate 10 -f image2 -i maze-tran-td-%03d.png maze-tran-td.gif
[...]
$
```

6 Homework

1. Solve the bottom-up case.
2. Play with different types of mazes! What if you actually draw and scan one yourself?

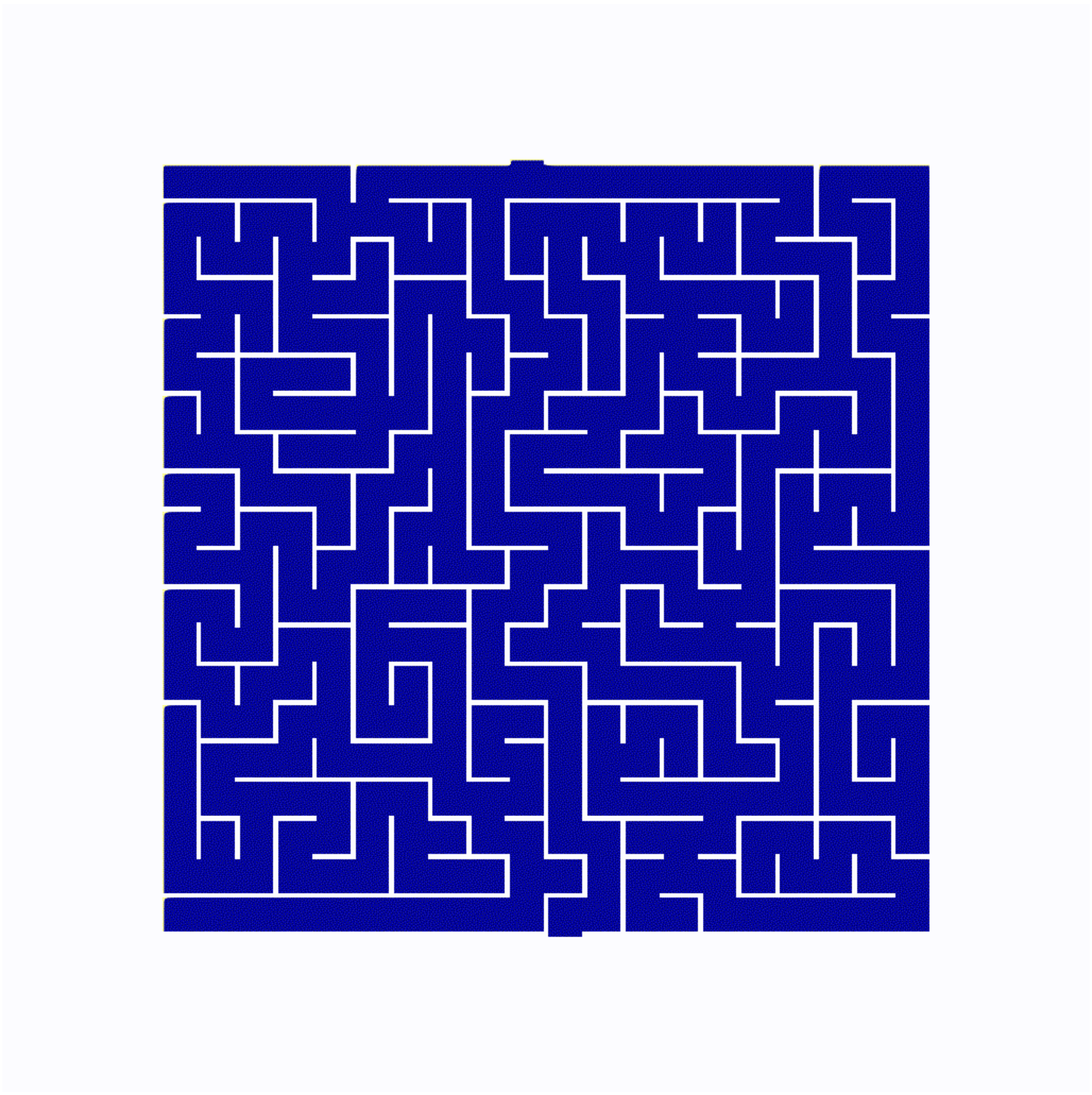
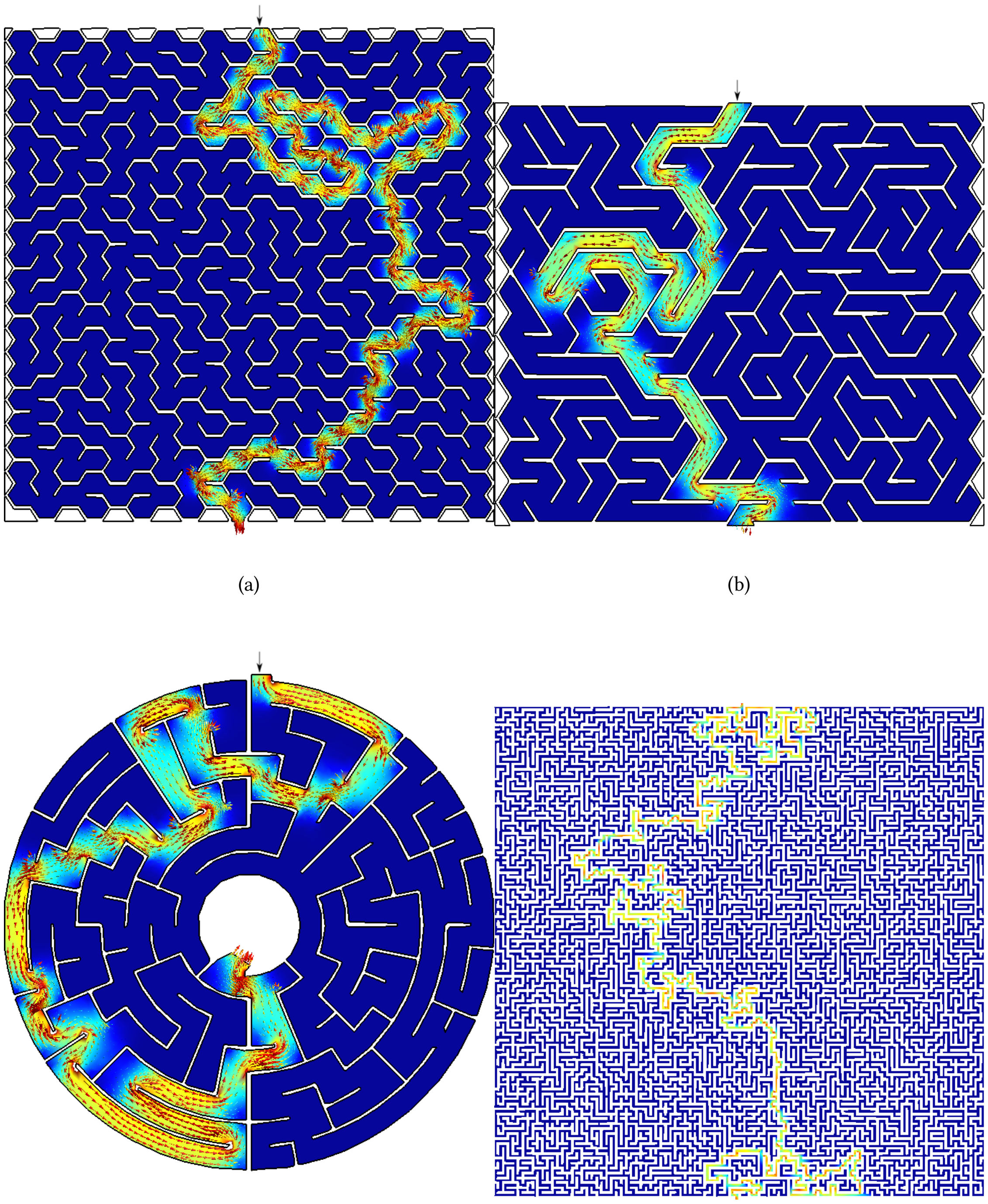


Figure 3: Maze solved as a transient Laplace problem with FeenoX



(a) (b)

(c) (d)

Figure 4: Any arbitrary maze (even hand-drawn) can be solved with FeenoX